

Infinite spin limit of semiclassical string states

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ABSTRACT: Motivated by recent works of Hofman and Maldacena and Dorey we consider a special infinite spin limit of semiclassical spinning string states in $AdS_5 \times S^5$. We discuss examples of known folded and circular 2-spin string solutions and demonstrate explicitly that the 1-loop superstring correction to the classical expression for the energy vanishes in the limit when one of the spins is much larger than the other. We also give a general discussion of this limit at the level of integral equations describing finite gap solutions of the string sigma model and argue that the corresponding asymptotic form of the string and gauge Bethe equations is the same.

KEYWORDS: Integrable Field Theories, AdS-CFT Correspondence, Bethe Ansatz.

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1. Introduction

In this paper, following recent work of [1–3], we explore a special limit of semiclassical string states in $AdS_5 \times S^5$ and dual gauge theory states in which one of the charges (one spin J in S^5) is much larger than all others. The energy (dimension) E diverges with J while their difference stays finite. This limit appears to bring in remarkable simplifications, and thus its study may help to further clarify the structure of the string/gauge spectrum of states.

If we consider for definiteness the $SU(2)$ sector or string states on $R \times S^3$ parametrized by the two angular momenta J_1, J_2 , then the limit we are interested in is, say, $J_2 \gg J_1$ and $E - J_2 = f(J_1, \lambda) + O(\frac{1}{J_2})$ (λ is the 't Hooft coupling or the square of the string tension). In the semiclassical approximation one assumes that $\lambda \gg 1$ while $\mathcal{J}_i = \frac{J_i}{\sqrt{\lambda}}$ are fixed. Taking this limit for a few known classical spinning string solutions [4–6] one finds that $E - J_2$ takes a simple “square root” form, and the analytic form of the solution simplifies. This turns out to be not accidental, as these states may be considered as bound states of “giant magnons” whose momentum is fixed in the large spin limit [2, 3]. Furthermore, their “square root” dispersion relation appears to be exact in λ , being protected by a residual supersymmetry in this limit [1, 2].

One of our aims below will be to confirm this explicitly by a 1-loop $AdS_5 \times S^5$ superstring theory computation. This is a non-trivial check as the presence and implications of the $SU(2|2) \times SU(2|2)$ (centrally extended) supersymmetry [1, 2] was not yet established directly at the level of the superstring action of [7]. We shall also supplement this by an analysis of the corresponding limit of the gauge/string Bethe equations of [8–10].

On the dual spin chain side this large spin limit corresponds to the large spin chain length $J = J_1 + J_2$, and the states for which $E - J$ is fixed for $J \rightarrow \infty$ are in the “intermediate” part of the spin chain spectrum. For example, at the leading 1-loop order in λ the spin chain spectrum has the following structure in the $J \rightarrow \infty$ limit [11] (the structure of the spectrum at finite λ is expected to be qualitatively similar): it starts with the ferromagnetic vacuum (BPS state) with $E - J = 0$, on top of which come magnon states with $E - J \sim \frac{\lambda}{J^2} + O(\frac{1}{J^3})$ dual to BMN states, then come low-energy spin wave states with $E - J \sim \frac{\lambda}{J} + O(\frac{1}{J^2})$ [12] dual to spinning strings [13, 5, 14], then “intermediate” states with $E - J \sim \lambda + O(\frac{1}{J})$ and finally the spinons and the top-energy antiferromagnetic state with $E - J \sim \lambda J + O(1)$.¹

While the momenta for standard magnon states, $p \sim \frac{n}{J}$ scale to zero with $J \rightarrow \infty$, the momenta of special elementary “giant magnon” states, a finite number of which are used to construct physical Bethe states in the “intermediate” part of the spectrum, are fixed in the large length limit. The same applies to the states in the near-antiferromagnetic region which are built out of an order J number of magnons. Indeed, the same limit was previously considered in [16] and, in particular, in [17–19] in connection with the antiferromagnetic state of the spin chain. The string solution counterpart of the antiferromagnetic state was found in [20].

Below in section 2 we shall describe the large spin limit of several classical string solutions on S^3 in S^5 (with the corresponding states belonging to the $SU(2)$ sector of the spin chain). One of them will be new — the second-spin generalization of the “giant magnon” of [2] (independently found recently in [21]) while two others will be special cases of the known solution – the folded spinning string of [6, 15] and the circular string of [5, 22]. In all of these cases we shall find that the expression for the classical energy simplifies in

¹The “microscopic” magnon states correspond to $J_2 \gg J_1$ with J_1 being finite; the “thermodynamic” limit which was used in [12, 15] to isolate the semiclassical spin wave states assumed that both J_1 and J_2 are large but their ratio J_1/J_2 (or “filling fraction”) is fixed. The present thermodynamic limit for semiclassical states corresponds to $J_2 \rightarrow \infty$ with $1 \ll J_1 \ll J_2$.

the limit $J_2 \rightarrow \infty$ and takes the universal form

$$E - J_2 = \sqrt{J_1^2 + \lambda k^2}, \tag{1.1}$$

where k is a constant depending on a particular solution. The same applies also to the circular (S, J) solution of [22] from the $SL(2)$ sector as we discuss in appendix B.

There are indications based on residual supersymmetry [2, 3] suggesting that semiclassical string solutions obtained in the above limit represent BPS states and thus their energy formula should not receive string $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections. In section 3 we shall compute the 1-loop string correction to the energies of folded and circular string solutions in the large J_2 limit using the methods of [13, 23]. On general grounds, the classical energy (1.1) of a classical solution may receive 1-loop string corrections of the form $E_1 = E_1(\mathcal{J}_1)$, $\mathcal{J}_1 = \frac{J_1}{\sqrt{\lambda}}$. We find that the 1-loop correction to the energy indeed vanishes in the $\mathcal{J}_2 \rightarrow \infty$, $\mathcal{J}_1 = \text{fixed}$ limit due to a nontrivial cancellation between the contributions of the bosonic and fermionic fluctuation modes. This suggests (like in the near-geodesic or plane-wave cases, cf. [24–26]), that here the superstring action expanded near the large-spin classical solution has a hidden world-sheet supersymmetry (a remnant of target-space supersymmetry after κ -symmetry gauge fixing), but so far it has not identified explicitly.²

In section 4 we shall return to the discussion of the large spin limit at the classical string level and present the general analysis of it using the integral equation [8] for the finite gap solutions of the string sigma model on S^3 . We shall then comment on the infinite length limit in the general Bethe ansatz equations on the gauge [9] and the string [10, 27, 28] sides and argue that they become the same in this limit, i.e. the “dressing factor” decouples.

In appendix A we discuss some technical details of the computation of 1-loop correction to the energy of 2-spin folded string solution in the $SU(2)$ sector.

The same large spin limit applies also to other sectors of states and we illustrate this on the example of the $SL(2)$ sector in appendices B and C and pulsating solutions in section 4.4. In appendix C we also consider giant magnons in the $SL(2)$ sector. It turns out that these magnons have infinite $E - J$ as well as an infinite Lorentz spin S . This is caused by the string reaching the boundary of AdS_5 . We show that there is a regularization that gives a finite answer and give a possible interpretation for this on the gauge side.

²One of the solutions for which we shall compute the 1-loop string correction will be the $J_1 = 0$ case of the $J_2 \rightarrow \infty$ limit of the folded string solution of [6], which is the same as the extremal limit of the single-spin folded string solution of [4]. Its classical energy $E - J_2 = 2\frac{\sqrt{\lambda}}{\pi}$ may be viewed as a $J_1 \rightarrow 0$ limit of $E - J_2 = \sqrt{J_1^2 + \frac{4\lambda}{\pi^2}}$ describing bound state of 2 giant magnons with spin [3]. In fact, the corresponding quantum state from the $SU(2)$ sector (i.e. the one dual to the BMN-type operator $\text{Tr}(Z \dots ZWZ \dots ZW \dots)$) should have $J_1 = 2$, not 0. At the level of the classical solution (obtained within the semiclassical expansion with $\lambda \gg 1$ and $\mathcal{J}_i = \frac{J_i}{\sqrt{\lambda}}$ fixed) one cannot of course distinguish between the $J_1 = 0$ and $J_1 = 2$ (or $J_1 = \text{any finite number}$) cases, but one may question what happens at the quantum level. Assuming that the relation $E - J_2 = \sqrt{J_1^2 + \frac{4\lambda}{\pi^2}}$ is exact and setting there $J_1 = 2$ we finish with $E - J_2 = 2\sqrt{1 + \frac{\lambda}{\pi^2}} = \frac{2\sqrt{\lambda}}{\pi} + 0 - \frac{\pi}{\sqrt{\lambda}} - \frac{\pi^3}{4(\sqrt{\lambda})^3} + \dots$. The absence of the 1-loop order $(\sqrt{\lambda})^0$ correction to the $J_1 = 0$ solution is thus also consistent with this exact square root formula.

2. Large spin limit of classical string solutions on $R \times S^3$

In this section we shall describe several classical string solutions in the infinite spin limit. We shall consider strings moving in S^3 part of S^5 in $AdS_5 \times S^5$

$$ds^2 = -dt^2 + d\theta^2 + \cos^2 \theta d\varphi_1^2 + \sin^2 \theta d\varphi_2^2 . \quad (2.1)$$

In general, a rigid rotating string configuration that we are interested in may be described as a solution of Nambu action in a “static” gauge

$$t = \tau , \quad \theta = \theta(\sigma) , \quad \varphi_1 = w_1 t + \tilde{\varphi}_1(\sigma) , \quad \varphi_2 = w_2 t + \tilde{\varphi}_2(\sigma) , \quad (2.2)$$

and thus carries the energy E and two angular momenta $J_i \sim w_i$.

2.1 “Giant magnons” with spin

The “giant magnon” solution considered in [2] was an open string with ends moving on a big circle³ which had $J_1 = 0$, $E, J_2 \rightarrow \infty$ with $E - J_2 = \frac{\sqrt{\lambda}}{\pi} \cos \theta_0$ finite. Here we shall generalize it to the case of finite non-zero J_1 , reproducing the energy formula first obtained on the spin chain side as the energy relation for a bound state of J_1 giant magnons in [3]⁴

$$E - J_2 = \sqrt{J_1^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} , \quad \sin \frac{p}{2} \equiv \cos \theta_0 . \quad (2.3)$$

The same classical solution was independently found in [21] using a relation to the sine-Gordon model.⁵ Setting

$$w_1 = w , \quad w_2 = 1 , \quad \tilde{\varphi}_1 = -w\psi(\sigma) , \quad \tilde{\varphi}_2 = \varphi(\sigma) , \quad (2.4)$$

the Lagrangian \mathcal{L} of the Nambu-Goto action $\mathcal{S} = \int d\tau \mathcal{L}$ is then determined to be

$$\mathcal{L} = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \sqrt{\mathcal{D}} , \quad (2.5)$$

where

$$\begin{aligned} \mathcal{D} = & (\dot{t}^2 - \cos^2 \theta \dot{\varphi}_1^2 - \sin^2 \theta \dot{\varphi}_2^2)[(\partial_\sigma \theta)^2 + \cos^2 \theta (\partial_\sigma \varphi_1)^2 + \sin^2 \theta (\partial_\sigma \varphi_2)^2] \\ & + (\cos^2 \theta \dot{\varphi}_1 \partial_\sigma \varphi_1 + \sin^2 \theta \dot{\varphi}_2 \partial_\sigma \varphi_2)^2 \end{aligned}$$

or, explicitly,

$$\mathcal{D} = \sin^2 \theta (\partial_\sigma \varphi)^2 + w^2 \cos^2 \theta (\partial_\sigma \psi)^2 + (1 - w^2) \cos^2 \theta (\partial_\sigma \theta)^2 - w^2 \sin^2 \theta \cos^2 \theta (\partial_\sigma \varphi + \partial_\sigma \psi)^2 . \quad (2.6)$$

³This solution is a also special case of string with spikes [29] on S^5 [30, 31].

⁴We interchange notation for J_1 and J_2 compared to [3].

⁵In conformal gauge, it can also be obtained as a solution of the generalized integrable Neumann model [32].

Varying \mathcal{L} with respect to ψ , we find the equation

$$\frac{\partial}{\partial \sigma} \left(\frac{-\cos^4 \theta \partial_\sigma \psi + \sin^2 \theta \cos^2 \theta \partial_\sigma \varphi}{\sqrt{\mathcal{D}}} \right) = 0, \quad (2.7)$$

which clearly has

$$\partial_\sigma \psi = \tan^2 \theta \partial_\sigma \varphi \quad (2.8)$$

as a special solution. Substituting (2.8) back into the action, we find the reduced action that determines the expression for θ as a function of φ

$$\mathcal{L} = \frac{\sqrt{1-w^2}\sqrt{\lambda}}{2\pi} \int d\varphi \sqrt{r^2 + r'^2}, \quad r \equiv \sin \theta, \quad r' \equiv \frac{dr}{d\varphi}. \quad (2.9)$$

Except for the extra $\sqrt{1-w^2}$ prefactor, eq. (2.9) is the same expression found in [2]; we thus get a “minimal” generalization of the “giant magnon” to the case of $w \sim J_1$ non-zero. The explicit form of the solution for θ is thus the same as in [2]

$$r = \sin \theta = \frac{\sin \theta_0}{\cos \varphi}, \quad -\frac{\pi}{2} + \theta_0 \leq \varphi \leq \frac{\pi}{2} - \theta_0. \quad (2.10)$$

θ then varies between $\frac{\pi}{2}$ and θ_0 . Then L reduces to

$$\mathcal{L} = \frac{\sqrt{\lambda}}{\pi} \sqrt{1-w^2} \sin \frac{p}{2}, \quad \sin \frac{p}{2} = \cos \theta_0, \quad (2.11)$$

where we have assumed that the momentum p of the magnon is related to θ_0 as in [2].

We can then derive from (2.6) and (2.9) the conserved quantities, the energy and the two spins,

$$\begin{aligned} E &= \frac{\sqrt{\lambda}}{2\pi} \int d\varphi \frac{r^2 + \frac{w^2 r^4 + r'^2}{1-r^2}}{\sqrt{(1-w^2)(r^2 + r'^2)}} \\ J_2 &= \frac{\sqrt{\lambda}}{2\pi} \int d\varphi \frac{r^2 \left(\frac{w^2 r^4 + r'^2}{1-r^2} + w^2 r^2 \right)}{\sqrt{(1-w^2)(r^2 + r'^2)}} \\ J_1 &= \frac{\sqrt{\lambda}}{2\pi} w \int d\varphi \frac{\sqrt{r^2 + r'^2}}{\sqrt{1-w^2}} = \frac{w}{1-w^2} \mathcal{L}. \end{aligned} \quad (2.12)$$

Here E and J are infinite, but their difference is finite and has the simple form

$$E - J_2 = \frac{\sqrt{\lambda}}{2\pi} \int d\varphi \frac{\sqrt{r^2 + r'^2}}{\sqrt{1-w^2}} = \frac{1}{1-w^2} \mathcal{L}. \quad (2.13)$$

Comparing (2.12) with (2.11), we find that

$$J_1 = \frac{w}{\sqrt{1-w^2}} \frac{\sqrt{\lambda}}{\pi} \sin \frac{p}{2}, \quad (2.14)$$

and hence from (2.13) we reproduce the energy formula (2.3).

To complete the solution, let us find the dependence of φ on ψ ; integrating (2.8) and using (2.10) and (2.14) gives

$$\varphi = \arctan(\cot \theta_0 \tanh(\cot \theta_0 \psi)) . \tag{2.15}$$

It is also convenient to express θ in terms of ψ

$$\theta = \arccos(\cos \theta_0 \operatorname{sech}(\cot \theta_0 \psi)) . \tag{2.16}$$

At the ends of the string, $\tan \varphi = \pm \cot \theta_0$, therefore $\psi \rightarrow \pm\infty$. In other words, the string wraps infinitely many times around the ψ or φ_1 direction. Note that as $\theta_0 \rightarrow 0$, $\varphi(\psi)$ approaches the step function $\varphi(\psi) = \frac{\pi}{2}\epsilon(\psi)$, while similarly $\theta(\psi)$ approaches $\theta(\psi) = \frac{\pi}{2}\epsilon(\psi)$. (We have continued θ to $\theta < 0$ since φ jumps by π as ψ changes sign). This behavior will be relevant when considering the folded string.

The discussion of finite gap solutions in section 4 below suggests that there should exist also more general solutions representing bound states of n magnons with total momentum p with energy

$$E - J_2 = \sqrt{J_1^2 + \frac{\lambda}{\pi^2} n^2 \sin^2 \frac{p}{2n}} = n \sqrt{\left(\frac{J_1}{n}\right)^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2n}} . \tag{2.17}$$

In the case of $J_1 = 0$ the special case of $\theta_0 = 0$ or $p = \pi$ and $n = 1$ corresponds to a string that stretches through the north pole of a 2-sphere [2]. A combination of $n = 2$ of such strings with total $p = 2\pi$ and thus with $E - J_2 = 2\sqrt{\frac{\lambda}{\pi}}$ is then a limit of a folded closed string rotating on S^2 with its center at rest at the north pole and the positions of the folds approaching the equator ($\theta = \frac{\pi}{2}$). Similarly, there exists an analogous $J_2 \rightarrow \infty$, $p = n\pi$ limit of the folded ($\frac{n}{2}$ times) 2-spin solution of [6, 15] with the simple energy formula found (for $n = 2$) in [3]⁶

$$E - J_2 = \sqrt{J_1^2 + \frac{\lambda}{\pi^2} n^2} . \tag{2.18}$$

We shall review this limit and present the explicit form of the resulting solution in the next subsection.

2.2 $J_2 \gg J_1$ limit of the folded string solution

Another example is found as a limit of the 2-spin folded string described in conformal gauge by the following ansatz (cf. (2.2), see also [14] for a review)

$$t = \kappa\tau, \quad \theta = \theta(\sigma), \quad \varphi_1 = w_1\tau, \quad \varphi_2 = w_2\tau, \tag{2.19}$$

where [6] ($\theta' \equiv \partial_\sigma \theta$)

$$\theta'' + \frac{1}{2} w_{21}^2 \sin 2\theta = 0, \quad w_{21}^2 \equiv w_2^2 - w_1^2 \tag{2.20}$$

⁶The $J_2 \gg J_1$ limit of the folded string solution of [6] was discussed (for $n = 2$) in appendix E in [15] where the leading term in the expansion of the square root at $J_1 > \sqrt{\lambda}$ was found.

where we assumed that $w_2 > w_1$ and for generality introduced the scaling parameter κ .⁷ Then

$$\theta'^2 = w_{21}^2 (\sin^2 \theta_* - \sin^2 \theta), \quad (2.21)$$

where θ_* determines the length of the folded string, i.e. $-\theta_* \leq \theta(\sigma) \leq \theta_*$. The conformal gauge constraint implies

$$\kappa^2 = \theta'^2 + w_1^2 \cos^2 \theta + w_2^2 \sin^2 \theta = w_1^2 \cos^2 \theta_* + w_2^2 \sin^2 \theta_*. \quad (2.22)$$

We shall consider the case of a single fold (the number of folds $\frac{n}{2}$ is easy to restore at any stage). The solution of (2.21) can be written in terms of the elliptic functions [6, 15]

$$\cos \theta(\sigma) = \text{dn}(w_{21}\sigma, q), \quad \sin \theta(\sigma) = \sqrt{q} \text{sn}(w_{21}\sigma, q). \quad (2.23)$$

$$q \equiv \sin^2 \theta_* = \frac{\kappa^2 - w_1^2}{w_2^2 - w_1^2}. \quad (2.24)$$

The periodicity in σ implies⁸

$$2\pi = \int_0^{2\pi} d\sigma = 4 \int_0^{\theta_*} \frac{d\theta}{w_{21} \sqrt{\sin^2 \theta_* - \sin^2 \theta}}, \quad w_{21} = \frac{2}{\pi} K(q). \quad (2.25)$$

The conserved charges are

$$E = \sqrt{\lambda} \kappa, \quad J_1 = \sqrt{\lambda} w_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2 \theta = \frac{2\sqrt{\lambda} w_1}{\pi w_{21}} \int_0^{\theta_*} \frac{\cos^2 \theta d\theta}{\sqrt{\sin^2 \theta_* - \sin^2 \theta}} \quad (2.26)$$

$$J_2 = \sqrt{\lambda} w_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2 \theta = \frac{2\sqrt{\lambda} w_2}{\pi w_{21}} \int_0^{\theta_*} \frac{\sin^2 \theta d\theta}{\sqrt{\sin^2 \theta_* - \sin^2 \theta}}. \quad (2.27)$$

The parameters w_1 , w_2 and θ_* can be determined in terms of J_1 and J_2 (and λ).⁹ Let us now follow [15, 3] and consider a special limit of this solution where $J_2 \gg J_1$, i.e. $\mathcal{J}_2 \rightarrow \infty$ for fixed \mathcal{J}_1 . As usual in a semiclassical expansion we assume that $\lambda \gg 1$ and $\mathcal{J}_1 = \frac{J_1}{\sqrt{\lambda}}$ is kept finite. It corresponds to the particular case when the string is maximally stretched in θ , so that its angular momentum J_2 around its centre of mass is maximal and goes to infinity (while the momentum of its center of mass J_1 is arbitrary).

Let us now take the limit $\theta_* \rightarrow \frac{\pi}{2}$, i.e. $q \rightarrow 1$. Let us distinguish two steps. First, the conformal constraint (2.22) implies that $w_2 = \kappa$. Second, the periodicity condition (2.25) leads to the conclusion that one must have $w_{21} \rightarrow \infty$. Indeed, in the limit $q \rightarrow 1$ we get $K(q) \rightarrow \infty$, so that

$$\theta_* \rightarrow \frac{\pi}{2}, \quad q \rightarrow 1, \quad \text{i.e.} \quad w_{21}, \kappa \rightarrow \infty. \quad (2.28)$$

⁷When $w_2 = w_1$ the solution is $\theta = m\sigma$, where m is an integer. This can be transformed [15] into the circular rotating solution with equal spins $J_1 = J_2$. In the limit when $\mathcal{J}_{1,2} = \infty$ it has $E = J_1 + J_2$, i.e. is equivalent to a BPS state represented by a point-like string.

⁸Here $K(q) \equiv \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-q \sin^2 \alpha}}$.

⁹Combining the above equations one obtains the two equations that determine $\mathcal{E} = \mathcal{E}(\mathcal{J}_1, \mathcal{J}_2)$, where $E = \sqrt{\lambda} \mathcal{E}$, $J_1 = \sqrt{\lambda} \mathcal{J}_1$, $J_2 = \sqrt{\lambda} \mathcal{J}_2$ [15]: $\left(\frac{\mathcal{E}}{K(q)}\right)^2 - \left(\frac{\mathcal{J}_1}{E(q)}\right)^2 = \frac{4}{\pi^2} q$, $\left(\frac{\mathcal{J}_2}{K(q)-E(q)}\right)^2 - \left(\frac{\mathcal{J}_1}{E(q)}\right)^2 = \frac{4}{\pi^2}$.

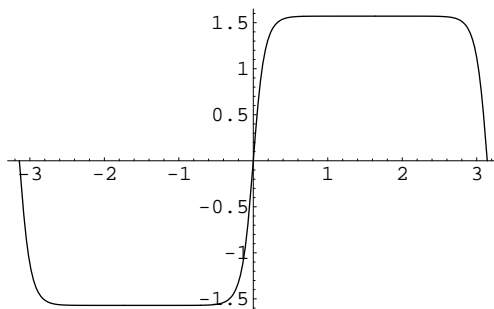


Figure 1: $\theta(\sigma) = \arcsin[\sqrt{q} \operatorname{sn}(w_{21}\sigma, q)]$, for $q = 0.99999999$, $-\pi \leq \sigma \leq \pi$.

If we do not impose the periodicity condition, we get a more general kink solution (see (2.34) below) which does not, however, represent a physical closed-string state.

Setting¹⁰

$$w_1 \equiv \kappa w, \quad w_{21} = \kappa \sqrt{1 - w^2}, \quad w < 1, \quad (2.29)$$

so that $\varphi_1 = wt$, $\varphi_2 = t$, cf. (2.4), we get from (2.21)¹¹

$$\theta' = \pm \kappa \sqrt{1 - w^2} \cos \theta. \quad (2.30)$$

To illustrate what happens as $\theta_* \rightarrow \frac{\pi}{2}$, i.e. as q approaches 1, one may plot the periodic solution $\theta(\sigma) = \arcsin[\sqrt{q} \operatorname{sn}(w_{21}\sigma, q)]$ with σ between $-\pi$ and π (see figure1). In the limit, $\theta(\sigma)$ for $-\pi < \sigma < \pi$ becomes just a *step function*, like the one considered previously, jumping from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. It can then be periodically extended to all σ , so that $\theta' \rightarrow \pm\infty$ at $\sigma = -\pi, 0, \pi, \dots$ and $\theta' \rightarrow 0$ at other points in agreement with (2.30). The energy of the solution and J_2 then approach infinity

$$E = \sqrt{\lambda} \kappa = \frac{2\sqrt{\lambda}}{\pi\sqrt{1-w^2}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos^2 \theta} \rightarrow \infty, \quad (2.31)$$

$$J_1 = \frac{2\sqrt{\lambda}}{\pi} \frac{w}{\sqrt{1-w^2}}, \quad J_2 = \frac{2\sqrt{\lambda}}{\pi\sqrt{1-w^2}} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin^2 \theta}{\cos^2 \theta} \rightarrow \infty, \quad (2.32)$$

while $E - J_2$ stays finite [3]

$$E - J_2 = \sqrt{J_1^2 + \frac{4\lambda}{\pi^2}}. \quad (2.33)$$

Let us mention that if one formally relaxes the periodicity condition in σ and introduces the new spatial variable $x = \kappa\sigma \in (-\infty, \infty)$ which will be fixed in the limit $\kappa \rightarrow \infty$ then the solution of (2.30) of the theory defined on a plane instead of a cylinder is

$$\theta(x) = \pm 2 \arctan \tanh\left(\frac{1}{2} \sqrt{1-w^2} x\right), \quad x \equiv \kappa\sigma. \quad (2.34)$$

¹⁰In order to have J_1 staying finite in the limit $\kappa \rightarrow \infty$ we need to rescale w_1 .

¹¹ $w = 1$ thus corresponds to the BPS limit when θ is constant.

This non-trivial solution (2.34) (which is not a limit of the periodic solution on a circle) appears only in the exact scaling limit and describes a kink localized near $x = 0$.¹²

Let us mention that for $w = 0$ eq.(2.34) represents a limit of the solution in [2] in the conformal gauge. The parameter θ_0 in [2] and in the previous subsection is formally related to θ_* by a $\frac{\pi}{2}$ shift. Indeed, here the center of the string is at the pole ($\theta = 0$) and its ends (at $\pm\theta_*$) approach the equator in the limit, while in the previous subsection the ends of the string were at the equator from the start while its center was approaching the pole as $\theta_0 \rightarrow 0$.

Another remark is that the energy formula (2.17) suggests the existence of more general closed string configurations with $J_1 = 0$ for which $p = 2M\pi$ with integer M

$$E - J_2 = \frac{\sqrt{\lambda}}{\pi} n \left| \sin \frac{M\pi}{n} \right|. \quad (2.35)$$

The corresponding closed string solution describes a string with spikes [29] on S^5 and was obtained in [31]. It has

$$\varphi_1 = 0, \quad \varphi_2 = \omega\tau + M\sigma, \quad \theta(\sigma) = \theta(\sigma + 2\pi). \quad (2.36)$$

In the limit $J_2 \rightarrow \infty$, one finds that $\omega \rightarrow 1$. For an arbitrary winding number M and number of cusps n , the closed string is built out of n segments with ends on the φ_2 -equator of S^2 (with minimal value of $\theta = \theta_*$ reached in the middle of each segment); all segments combine to cover the $2\pi M$ distance along the equator. For $M = 1$, $n = 2$, one recovers the folded string, or more generally, for $M = \frac{n}{2}$ one gets $\frac{n}{2}$ -folded string solution for which the string stretches between the opposite points on the equator passing through the north pole in θ (i.e. in this case $\theta_* = \frac{\pi}{2}$).

2.3 $J_2 \gg J_1$ limit of circular string solution

The simplest circular 2-spin string solution on S^3 is represented in conformal gauge by [22] (cf. (2.1),(2.2),(2.19))

$$t = \kappa\tau, \quad \theta = \theta_0 = \text{const}, \quad \varphi_1 = w_1\tau + m_1\sigma, \quad \varphi_2 = w_2\tau + m_2\sigma. \quad (2.37)$$

Written in terms of 2 complex combinations of embedding coordinates of S^3 into R^4 we have

$$X_1 = a_1 e^{iw_1\tau + im_1\sigma}, \quad X_2 = a_2 e^{iw_2\tau + im_2\sigma}, \quad |a_1|^2 + |a_2|^2 = 1, \quad (2.38)$$

where $a_1 = \cos \theta_0$, $a_2 = \sin \theta_0$. The energy and two spins are

$$E = \sqrt{\lambda}\mathcal{E}, \quad J_i = \sqrt{\lambda}\mathcal{J}_i, \quad \mathcal{E} = \kappa, \quad \mathcal{J}_i = a_i^2 w_i, \quad (2.39)$$

where the equations of motion and conformal gauge conditions imply ($i = 1, 2$)

$$w_i = \sqrt{m_i^2 + \nu^2}, \quad \kappa^2 = 2 \sum_i a_i^2 w_i^2 - \nu^2, \quad \sum_i a_i^2 w_i m_i = 0. \quad (2.40)$$

¹²This solution of the sin-Gordon equation may be interpreted as describing a zero-energy particle that goes from one maximum of the $-\cos^2 \theta$ potential to another in an infinite amount of “time” x (we have $\theta(0) = 0$, $\theta(x = \pm\infty) = \pm\frac{\pi}{2}$).

This gives

$$\mathcal{E}^2 = 2 \sum_i \sqrt{m_i^2 + \nu^2} \mathcal{J}_i - \nu^2, \quad \sum_i m_i \mathcal{J}_i = 0, \quad \sum_i \frac{\mathcal{J}_i}{\sqrt{m_i^2 + \nu^2}} = 1. \quad (2.41)$$

Here we are interested in the solution when $J_2 \gg J_1$. To consider this it is useful to fix one of the two winding numbers to be 1 (it is easy to restore its general value at the end); setting

$$m_2 = 1, \quad m_1 = -m, \quad \mathcal{J}_2 = m \mathcal{J}_1 \quad (2.42)$$

we should thus expand the above relations in large m at fixed \mathcal{J}_1 . In general, the relation between the spins and the energy is found by eliminating ν from the following two equations

$$\frac{m \mathcal{J}_1}{\sqrt{1 + \nu^2}} + \frac{\mathcal{J}_1}{\sqrt{m^2 + \nu^2}} = 1, \quad \mathcal{E}^2 = 2\sqrt{1 + \nu^2} m \mathcal{J}_1 + 2\sqrt{m^2 + \nu^2} \mathcal{J}_1 - \nu^2. \quad (2.43)$$

Expanding in large m we get from the first equation

$$\nu^2 = m^2 \mathcal{J}_1^2 + \frac{2\mathcal{J}_1^3}{\sqrt{1 + \mathcal{J}_1^2}} m + \mathcal{J}_1^2 - \frac{1 + 3\mathcal{J}_1^2}{(1 + \mathcal{J}_1^2)^2} + \frac{\mathcal{J}_1^3(1 + 6\mathcal{J}_1^2)}{(\sqrt{1 + \mathcal{J}_1^2})^7} \frac{1}{m} + O\left(\frac{1}{m^2}\right) \quad (2.44)$$

Then the second equation in (2.43) gives

$$\mathcal{E} = \kappa = m \mathcal{J}_1 + \sqrt{1 + \mathcal{J}_1^2} - \frac{1}{2m} \frac{\mathcal{J}_1}{1 + \mathcal{J}_1^2} + O\left(\frac{1}{m^2}\right), \quad (2.45)$$

so that in the strict $m \rightarrow \infty$, $\kappa \rightarrow \infty$ limit we get (recalling that $\mathcal{J}_2 = m \mathcal{J}_1$)

$$E - J_2 = \sqrt{J_1^2 + \lambda}. \quad (2.46)$$

This is similar to the expressions (2.3),(2.33) found above for other solutions in the same limit.

Let us comment on the form of the limiting solution. In the limit the string becomes infinitely long (has infinite winding number m_1) but has infinitesimal radius and its position approaches $\theta_0 = \frac{\pi}{2}$. One can formally express the limiting solution in terms of the coordinates on $R \times R$ instead of $R \times S^1$ which one may keep finite in the limit $\kappa \rightarrow \infty$, $\mathcal{J}_2 \rightarrow \infty$. For $m_2 = 1$ we get:

$$X_1 = a_1 e^{i\sqrt{1 + \mathcal{J}_1^2} t - i\mathcal{J}_1^{-1} x}, \quad X_2 = a_2 e^{it}, \quad t = \kappa \tau, \quad x = \kappa \sigma, \quad (2.47)$$

where the limiting values of the parameters a_i are¹³

$$a_1 \approx \frac{\mathcal{J}_1}{(1 + \mathcal{J}_1^2)^{1/4}} \frac{1}{\sqrt{\mathcal{J}_2}} \rightarrow 0, \quad a_2 \approx 1 - \frac{\mathcal{J}_1^2}{2\sqrt{1 + \mathcal{J}_1^2}} \frac{1}{\mathcal{J}_2} \rightarrow 1. \quad (2.48)$$

Restoring the dependence on the second winding number $m_2 \equiv k$ we get

$$E - J_2 = \sqrt{J_1^2 + \lambda k^2}. \quad (2.49)$$

A similar limit exists for a circular (S, J) string in the $SL(2)$ sector [22]; we discuss this in appendix B.

¹³In general, for $m_2 = 1$ the constants a_1, a_2 can be expressed as [22] $a_1^2 = \frac{\sqrt{1 + \nu^2}}{m\sqrt{m^2 + \nu^2} + \sqrt{1 + \nu^2}}$, $a_2^2 = \frac{m\sqrt{m^2 + \nu^2}}{m\sqrt{m^2 + \nu^2} + \sqrt{1 + \nu^2}}$.

3. 1-loop correction to the energy of folded and circular string in the $J_2 \rightarrow \infty$ limit

In this section we shall perform a check of the exactness of the energy formulae for the folded (2.33) and circular (2.46) solutions by computing their 1-loop string corrections and showing that they vanish.

3.1 Folded string case

In appendix A we have presented some details of the computation of the bosonic and fermionic quadratic fluctuation actions near the folded string solution (2.19) for arbitrary J_1, J_2 , i.e. arbitrary parameter θ_* . Here we shall specialize to the limiting case of interest (2.28): $\theta_* = \frac{\pi}{2}$, $\kappa \rightarrow \infty$.

Before getting into the more technical details of the computation let us sketch some of its general features. For finite κ the 1-loop correction to the energy is given by the sum over characteristic frequencies, i.e., symbolically,

$$E_1 = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \sum_r c_r \sqrt{n^2 + M_r^2}, \tag{3.1}$$

where c_r are multiplicity and sign factors, n is the discrete momentum on a circle $\sigma \in (-\pi, \pi)$ and M_r are effective masses depending on parameters of the solution. The $\frac{1}{\kappa}$ factor is the proportionality coefficient between the space-time and 1-d energy reflecting that $t = \kappa\tau$. In the large κ limit M_r will scale as $M_r \rightarrow \kappa\bar{M}_r$; introducing $p_n = \frac{n}{\kappa}$ and keeping only the leading order in $\kappa \rightarrow \infty$ one can then replace the sum over n by an integral over a continuous momentum variable conjugate to spatial variable $x = \kappa\sigma$ (see also [13] for a discussion of a similar limit):

$$E_1 = \frac{1}{2} \int_{-\infty}^{\infty} dp \sum_r c_r \sqrt{p^2 + \bar{M}_r^2} + O\left(\frac{1}{\kappa}\right). \tag{3.2}$$

The same result can be arrived at directly by introducing the κ -rescaled variables as in (2.29) $w_1 = \kappa w$, $t = \kappa\tau$, $x = \kappa\sigma$. Then the resulting quadratic fluctuation action can be written as $S = \int dt \int_{-\infty}^{\infty} dx \bar{L}$.

In computing \bar{L} and thus \bar{M}_r for the present case of the folded solution we should remember to use the form of the solution as it appears in the large κ limit of the original periodic solution on a σ -circle, and not the formal solution on an infinite line (2.34) that exists in the strict scaling limit. In other words, $\theta(\sigma)$ should be replaced by a periodic version of the step function $\frac{\pi}{2}\epsilon(\sigma)$ which is a large κ limit of the solution (2.23).

Let us now consider in turn the relevant bosonic and fermionic fluctuations as they appear in \bar{L} . The AdS_5 fluctuations in (4.1) have rescaled mass equal to 1, and the masses of two decoupled S^5 fluctuations in (4.6),(4.4) are given by

$$\bar{M}_3^2 = -\bar{\Lambda} = 1 - 2(1 - w^2) \cos^2 \theta, \quad \Lambda = \kappa^2 \bar{\Lambda}. \tag{3.3}$$

The Lagrangian for the remaining three S^5 bosonic fluctuations (4.11) takes the form (here $f' = \partial_x f$, $\dot{f} = \partial_t f$)

$$\bar{L} = \frac{1}{2} \left[\dot{\eta}^2 + \dot{f}_1^2 + \dot{f}_2^2 - \eta'^2 - f_1'^2 - f_2'^2 - \bar{M}_1^2(\eta^2 + f_1^2) - \bar{M}_2^2 f_2^2 + 4(w \sin \theta f_1 - \cos \theta f_2) \dot{\eta} \right], \quad (3.4)$$

$$\bar{M}_1^2 = (w^2 - 1) \cos 2\theta, \quad \bar{M}_2^2 = (w^2 - 1)(1 + \cos 2\theta). \quad (3.5)$$

As already mentioned above, $\theta(\sigma)$ should be replaced by the periodic extension of the step function $\frac{\pi}{2}\epsilon(\sigma)$ at $-\pi < \sigma < \pi$. To leading order in large κ one may formally replace it by $\frac{\pi}{2}\epsilon(x)$ ¹⁴

$$\theta(x) = \frac{\pi}{2}\epsilon(x), \quad \epsilon(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \quad (3.6)$$

Thus θ is essentially constant for $x > 0$ and for $x < 0$ (i.e. the string is close to a point-like geodesic state).¹⁵ Then

$$\sin \theta = \epsilon(x), \quad \cos \theta = 1 - \epsilon^2(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}. \quad (3.7)$$

If we ignore the contribution of the point $x = 0$,¹⁶ we find that the mass (3.3) of the two decoupled S^5 fluctuations becomes equal to 1, and that f_2 in (3.4) becomes massless and decouples. We are left with the following Lagrangian for η and f_1

$$\bar{L} = \frac{1}{2} \left[\dot{\eta}^2 - \eta'^2 + \dot{f}_1^2 - f_1'^2 + (w^2 - 1)(\eta^2 + f_1^2) + 4w\epsilon(x)f_1\dot{\eta} \right]. \quad (3.8)$$

Using that $\epsilon^2 = 1$ away from the point $x = 0$, we end up with the following characteristic frequencies (conjugate to time variable t)

$$\omega = \pm w \pm \sqrt{p^2 + 1}, \quad (3.9)$$

where p is a continuous 1-dimensional momentum corresponding to the x -direction. We explain the derivation of (3.9) in detail at the end of appendix A.

Let us now consider the fermionic fluctuations in (4.16),(4.20), where we set $w_2 = \kappa$ and rescale the coordinates by κ . We shall also use that (2.30) implies $\theta' = \pm\sqrt{1 - w^2} \cos \theta$, $\theta'' = -(1 - w^2) \sin \theta \cos \theta$, where here and below prime stands for ∂_x (and dot for ∂_t). To

¹⁴More precisely, one needs also to include step functions at $\pm\infty$. It turns out that contributions of isolated points, such as $x = 0, \pm\infty$, may be ignored when computing the spectrum.

¹⁵This limit of the folded solution written in cartesian coordinates is $X_1 = [1 - \epsilon^2(x)]e^{iwt}$, $X_2 = \epsilon(x)e^{it}$, so that the size of the string shrinks to zero in X_1 plane apart from $x = 0$ (and $x = \pm\infty$). This is similar to what was found in the case of the circular solution (2.47).

¹⁶A qualitative reason why one can ignore the contribution of this single point is that we are computing an extensive quantity and the coefficient function in the corresponding differential equation for the fluctuations is finite at this point (i.e. this is different from, e.g., a delta-function potential case).

simplify the fermionic operator D_F in (4.20) we perform the rotations in the (89) and (08)-planes:

$$\vartheta = e^{\frac{1}{2}s\Gamma_8\Gamma_9} e^{\frac{1}{2}v\Gamma_0\Gamma_8} \tilde{\vartheta}, \quad \sin s = \frac{1}{u} \sin \theta, \quad \cos s = \frac{w}{u} \cos \theta, \quad (3.10)$$

$$u = \tanh v = \sqrt{\sin^2 \theta + w^2 \cos^2 \theta}, \quad \cosh v = \frac{1}{\sqrt{1 - w^2 \cos^2 \theta}}. \quad (3.11)$$

Then D_F becomes

$$\begin{aligned} D_F &= \Gamma_0 \sqrt{1 - w^2} \cos \theta \partial_t - \theta' \Gamma_7 \partial_x + u \theta' \Gamma_{078} \Gamma_{1234} \\ &+ \frac{\sqrt{1 - w^2}}{2u} \Gamma_0 \cos \theta \left[\sqrt{1 - w^2} \sin \theta (-u \Gamma_0 + \Gamma_8) + w \Gamma_9 \right] \Gamma_7 \\ &+ \theta'^2 \left(\frac{1}{2u} \tan \theta \Gamma_0 - \frac{w}{2u^2 \sqrt{1 - w^2} \cos \theta} \Gamma_9 \right) \Gamma_{78} + \theta'^2 \frac{w}{2u \sqrt{1 - w^2} \cos \theta} \Gamma_{709}. \end{aligned} \quad (3.12)$$

If we further do a rescaling of the fermionic variable, introducing

$$\Theta = \sqrt{\theta'} \tilde{\vartheta}, \quad L_F = -2i\kappa \bar{\Theta} \hat{D}_F \Theta, \quad (3.13)$$

we obtain

$$\hat{D}_F = \pm \Gamma_0 \partial_t - \Gamma_7 \partial_x \mp \frac{w}{2u^2} \Gamma_{789} + u \Gamma_{078} \Gamma_{1234}, \quad (3.14)$$

where the upper signs correspond to $x < 0$, while the lower sign to $x > 0$ (they come from $\partial_x \theta = \pm \sqrt{1 - w^2} \cos \theta$). Since $\Gamma_{1234}^2 = 1$, we can restrict to subspaces satisfying $\Gamma_{1234} \Theta = \pm \Theta$.

Let us now specialize to the relevant case when θ is replaced by the step-function (3.6). Ignoring again the contribution of the $x = 0$ point and using that then $u = 1$ for $x < 0$, and $u = -1$ for $x > 0$, we get

$$\hat{D}_F = \pm \Gamma_0 \partial_t - \Gamma_7 \partial_x \mp \frac{w}{2} \Gamma_{789} \pm \Gamma_{078}. \quad (3.15)$$

Computing the determinant of this operator (now having constant coefficients), and solving the resulting characteristic equations on either side of $x = 0$, one finds that the corresponding frequencies are similar to (3.9), i.e. they are essentially the BMN ones up to a w -dependent shift,

$$\omega = \pm \frac{w}{2} \pm \sqrt{p^2 + 1}. \quad (3.16)$$

Combining the contributions of all modes to the 1-loop shift of the energy (taking into account proper sign factors in (4.21) implying that the w -dependent shifts in (3.9) and (3.16) drop out) one finds that, just as in the BMN case, the 8 non-trivial bosonic mode contributions cancel against the 8 fermionic contributions, therefore, the 1-loop correction to the energy vanishes,

$$E_1 = 0. \quad (3.17)$$

3.2 Circular string case

Let us now perform a similar computation in the case of the large spin limit of the circular solution discussed in section 2.3. The bosonic fluctuation Lagrangian near the circular solution with generic $\mathcal{J}_1, \mathcal{J}_2$ was found in [22]. In addition to 4 AdS_5 massive fluctuations with mass κ there are 2 free fluctuations (corresponding to the X_3 direction of S^5) which have mass ν . Using (2.44) and rescaling the coordinates by κ as above, we end up with the corresponding characteristic frequencies, given in the $\kappa \rightarrow \infty$ limit by the same expression

$$\omega = \pm \sqrt{p^2 + 1}. \quad (3.18)$$

The remaining 3 coupled S^5 fluctuations in general are described by the following Lagrangian [22]

$$L = \frac{1}{2}(f_1'^2 + f_2'^2 + g_2'^2 - f_1'^2 - f_2'^2 - g_2'^2) + 2(a_2 w_1 f_1 - a_1 w_2 f_2) \dot{g}_2 - 2(a_2 m_1 f_1 - a_1 m_2 f_2) g_2' \quad (3.19)$$

Setting $m_2 = 1$, $m_1 = -m$ and rescaling the world-sheet coordinates by $\kappa = m\mathcal{J}_1 \rightarrow \infty$ (see (2.45)) we end up with the following analog of (3.4)

$$\bar{L} = \left[\frac{1}{2}(f_1'^2 + f_2'^2 + g_2'^2 - f_1'^2 - f_2'^2 - g_2'^2) + 2g_2 f_1 \sqrt{1 + \gamma^2} + 2\gamma g_2' f_1 \right], \quad (3.20)$$

$$\gamma \equiv \mathcal{J}_1^{-1}. \quad (3.21)$$

f_2 thus decouples in the limit and becomes massless. The non-trivial characteristic frequencies are then found to be

$$\omega_{1,2} = \sqrt{1 + \gamma^2} \pm \sqrt{(p + \gamma)^2 + 1}, \quad \omega_{3,4} = -\sqrt{1 + \gamma^2} \pm \sqrt{(p - \gamma)^2 + 1}. \quad (3.22)$$

Interestingly, while the circular solution is unstable at finite \mathcal{J}_2 [22], it becomes stable in the present limit, i.e. all characteristic frequencies are real.

The fermionic fluctuation Lagrangian for the general circular solution with two unequal spins was found in [33] (see also [34]). In the notation of [33]

$$L = 2i \bar{\vartheta} D_F \vartheta, \quad D_F = \begin{pmatrix} \Delta_F^+ & 0 \\ 0 & \Delta_F^- \end{pmatrix} \otimes 1 \quad (3.23)$$

$$\Delta_F^\pm = \bar{\sigma}^a \partial_a \mp W \bar{\sigma}^{012} \mp Q \bar{\sigma}^{134}, \quad (3.24)$$

where $\bar{\sigma}^\mu, \sigma^\mu$ are 16×16 gamma matrices in ten dimensions and $a = 0, 1$. Here

$$W^2 = a_1^2(m_1^2 + \nu^2) + a_2^2(m_2^2 + \nu^2), \quad M^2 = a_1^2 m_1^2 + a_2^2 m_2^2, \quad Q = \frac{a_1 a_2}{2MW} \kappa(m_1^2 - m_2^2). \quad (3.25)$$

One can compute the characteristic frequencies from the following determinant

$$\det \Delta_F^\pm = (\partial_0^2 - \partial_1^2)^2 + 2W^2(\partial_0^2 - \partial_1^2) + 2Q^2(\partial_0^2 + \partial_1^2) + (Q^2 + W^2)^2 = 0, \quad (3.26)$$

In the large κ limit one finds

$$W^2 = \kappa^2 + \dots, \quad M^2 = \frac{\kappa\gamma}{\sqrt{1 + \gamma^2}} + \dots, \quad Q^2 = \frac{1}{4}\gamma\kappa + \dots \quad (3.27)$$

After the rescaling of world-sheet coordinates we get from $\det\Delta_F^\pm = 0$ the following fermionic characteristic frequencies (with 4-fold degeneracy)

$$\omega = \pm\sqrt{(p \pm \frac{1}{2}\gamma)^2 + 1} . \tag{3.28}$$

Collecting the resulting bosonic and fermionic frequencies and observing that after the rescaling of τ by κ the 2d and space-time energies are the same, we finish with the following expression for the 1-loop correction to the energy¹⁷

$$E_1 = \frac{1}{2} \int_{-\infty}^{\infty} dp \left[6\sqrt{p^2 + 1} + \sqrt{(p + \gamma)^2 + 1} + \sqrt{(p - \gamma)^2 + 1} - 4\sqrt{(p + \frac{1}{2}\gamma)^2 + 1} - 4\sqrt{(p - \frac{1}{2}\gamma)^2 + 1} \right] . \tag{3.29}$$

This integral is convergent, and evaluating it directly one finds that it vanishes,

$$E_1 = 0 . \tag{3.30}$$

It is interesting to note that this vanishing is due to a non-trivial cancellation between the fermionic and bosonic contributions. Indeed, if we shift the fermions momentum in (3.29) by r , the resulting integral is still convergent,

$$I(\gamma, r) \equiv \frac{1}{2} \int_{-\infty}^{\infty} dp \left[6\sqrt{p^2 + 1} + \sqrt{(p + \gamma)^2 + 1} + \sqrt{(p - \gamma)^2 + 1} - 4\sqrt{(p + r)^2 + 1} - 4\sqrt{(p - r)^2 + 1} \right] = \gamma^2 - 4r^2 . \tag{3.31}$$

However, it vanishes only if $r = \frac{1}{2}\gamma$ as in (3.29), suggesting the presence of hidden $2d$ supersymmetry in this problem.

The generalization of the above expressions to the case of non-trivial second winding number $m_2 = k$ can be found by replacing $\gamma = \mathcal{J}_1^{-1} \rightarrow k\mathcal{J}_1^{-1}$; this does not change the conclusion about the vanishing of the 1-loop correction to the energy in this limit.

4. Infinite spin limit and bound magnons in integral Bethe equations

In [8] it was shown how to generate classical solutions for strings propagating on $R \times S^3$ and compare the results to gauge theory predictions using finite gap equations. In this section we will discuss the scaling limit and solutions of [2, 3, 21] and section 2 using this formalism.

This will then allow us, in particular, to argue that gauge theory and string theory predictions should match in this limit.

¹⁷Upon using the signs factors from (4.23) for the contributions of the frequencies (3.22) one finds that the p -independent parts of them cancel out.

4.1 Classical finite gap equations for a string on $R \times S^3$

Let us first summarize the results of [8]. The string sigma model action on $R \times S^3$ in conformal gauge can be written as

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[-(\partial_a t)^2 + \frac{1}{2} \text{Tr}(j_a^2) \right], \quad (4.1)$$

where j_a are the right currents which are written in terms of the $SU(2)$ group element \mathcal{G} as $j_a = \mathcal{G}^{-1} \partial_a \mathcal{G} = \frac{1}{2i} j_a^A \sigma^A$. The equations of motion that follow from (4.1) are

$$\partial_+ j_- + \partial_- j_+ = 0, \quad \partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0, \quad \partial_+ \partial_- t = 0, \quad . \quad (4.2)$$

We can also define the left currents $l_a = \mathcal{G} j_a \mathcal{G}^{-1} = \partial_a \mathcal{G} \mathcal{G}^{-1}$. The charges coming from the third component of the left and right currents are

$$Q_L^3 = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma l_0^3 = J_2 + J_1, \quad Q_R^3 = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma j_0^3 = J_2 - J_1. \quad (4.3)$$

A solution for t in (4.2) is $t = \kappa\tau$, and so the string energy E is given by

$$E = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \partial_\tau t = \sqrt{\lambda} \kappa. \quad (4.4)$$

We can now set up a pair of linear equations that are satisfied provided the string equations of motion are satisfied:

$$\begin{aligned} \left[\partial_\sigma + \frac{g}{\sqrt{2}} \left(\frac{j_+}{\frac{g}{\sqrt{2}} - x} - \frac{j_-}{\frac{g}{\sqrt{2}} + x} \right) \right] \Psi &= 0, \\ \left[\partial_\tau + 2\pi \frac{g}{\sqrt{2}} \left(\frac{j_+}{\frac{g}{\sqrt{2}} - x} + \frac{j_-}{\frac{g}{\sqrt{2}} + x} \right) \right] \Psi &= 0, \quad g^2 \equiv \frac{\lambda}{8\pi^2}. \end{aligned} \quad (4.5)$$

where x is a spectral parameter (not to be confused with the spatial coordinate used in the previous sections). The first equation can be integrated to give the monodromy matrix (given by path-ordered product)

$$\Omega(x) = \text{P exp} \int_0^{2\pi} d\sigma \frac{g}{2\sqrt{2}} \left(\frac{j_+}{\frac{g}{\sqrt{2}} - x} - \frac{j_-}{\frac{g}{\sqrt{2}} + x} \right). \quad (4.6)$$

Because of its unimodularity $\Omega(x)$ has eigenvalues $e^{\pm i P(x)}$ and satisfies the equation

$$\text{Tr} \Omega(x) = 2 \cos P(x), \quad (4.7)$$

where $P(x)$ is the quasi-momentum. It is clear from the Virasoro constraints

$$\frac{1}{2} \text{Tr} j_+^2 = \frac{1}{2} \text{Tr} j_-^2 = -\kappa^2, \quad (4.8)$$

and (4.6) that $P(x)$ has the pole structure

$$P(x) = -\frac{E/4}{x \pm \frac{g}{\sqrt{2}}} + \dots \quad (x \rightarrow \mp \frac{g}{\sqrt{2}}). \quad (4.9)$$

The asymptotic properties of $P(x)$ are determined by the charges Q_L and Q_R . For large x , $P(x)$ behaves as

$$P(x) = -\frac{J_2 - J_1}{2x} + \dots \quad (x \rightarrow \infty). \quad (4.10)$$

For small x , using $\Omega(0) = 1$ and expanding about $x = 0$, one finds

$$P(x) = 2\pi m + \frac{J_2 + J_1}{2} x + \dots \quad (x \rightarrow 0). \quad (4.11)$$

Here m is an integer, which follows from the periodicity condition in σ for a closed string. We will refer to $2\pi m$ as the string momentum, and this can be thought of as a level matching condition on the string.

Since $P(x)$ is not single valued, there can be an interesting singularity structure in the x complex plane. There are two types of singularities that we can have. First, there can be branch cuts along contours \mathcal{C}_k where two eigenvalues of the monodromy matrix are interchanged on either side of the cut, up to a factor of 2π . Hence,

$$P(x + i0) + P(x - i0) = 2\pi n_k, \quad x \in \mathcal{C}_k. \quad (4.12)$$

We can also have singular points in the complex plane such that $P(x)$ jumps by a multiple of 2π when transported around the singularity. These singularities will pair up such that $P(x)$ jumps by a multiple of 2π when it crosses a contour between the two singularities. We call this contour a condensate and label condensate j by \mathcal{B}_j .

Because of the cuts \mathcal{C}_k , the spectral parameter space becomes a two-sheeted surface, with the singularities in (4.9) appearing on both sheets. It is convenient to define the resolvent $G(x)$

$$G(x) = P(x) + \frac{E/4}{x + \frac{g}{\sqrt{2}}} + \frac{E/4}{x - \frac{g}{\sqrt{2}}}, \quad (4.13)$$

which is free of these poles on the top sheet. Hence, on this *physical* sheet, $G(x)$ can be expressed as

$$G(x) = \sum_k \int_{\mathcal{C}_k} dx' \frac{\rho(x')}{x - x'} + \sum_j \int_{\mathcal{B}_j} dx' \frac{\rho(x')}{x - x'}, \quad (4.14)$$

where $\rho(x')$ acts as a density along the cuts and condensates. The density along a condensate is readily determined to be $\rho(x') = -in_j$ if $x' \in \mathcal{B}_j$. Along the cuts, the condition in (4.12) can be reformulated as an integral equation for the density

$$G(x + i0) + G(x - i0) = 2 \int dx' \frac{\rho(x')}{x - x'} = \frac{xE}{x^2 - g^2/2} + 2\pi n_k, \quad x \in \mathcal{C}_k. \quad (4.15)$$

The asymptotic behavior for large and small x in (4.10) and (4.11) leads to the conditions

$$\begin{aligned} \int dx \rho(x) &= J_1 + \frac{E - J_2 - J_1}{2}, \\ \int dx \frac{\rho(x)}{x} &= 2\pi m, \\ \int dx \frac{\rho(x)}{x^2} &= \frac{E - J_2 - J_1}{g^2}. \end{aligned} \quad (4.16)$$

We can then rewrite the integral equation in (4.15) in terms of the inputs J_1 and J_2 as

$$2 \int dx' \frac{\rho(x')}{x-x'} = \frac{x(J_1+J_2)}{x^2-g^2/2} + g^2 x \int dx' \frac{\rho(x')}{x'^2(x^2-g^2/2)} + 2\pi n_k, \quad x \in \mathcal{C}_k. \quad (4.17)$$

This integral equation [8] is normally the main tool for finding string solutions, but we will see that it is not relevant for solutions made up only of giant magnons!

4.2 Infinite J limit and matching to asymptotic spin chain Bethe equations

Eq.(4.17) can be compared to the integral equation that follows in the “thermodynamic” ($J_1, J_2 \gg 1$, $\frac{J_1}{J_2}$ =fixed) limit [5, 12] from the proposed asymptotic Bethe ansatz on the gauge theory side [9]

$$2 \int dx' \frac{\rho(x')}{x-x'} = \frac{x(J_1+J_2)}{x^2-g^2/2} + g^2 x \int dx' \frac{\rho(x')}{xx'(xx'-g^2/2)} + 2\pi n_k, \quad x \in \mathcal{C}_k. \quad (4.18)$$

In general, the two equations (4.17) and (4.18) do not match starting with “3-loop” order implying the need to introduce an extra “dressing factor” into the spin chain Bethe ansatz [10].

If we now consider the scaling limit in which E and J_2 become infinite, but their difference $E - J_2$ as well as J_1 stay finite, then it follows from (4.16) that the second term in the r.h.s. of (4.17) which was the cause of difference between (4.17) and (4.18) is vanishingly small compared to the first term. This also implies that the l.h.s of (4.17) is negligible, and hence n_k must be infinite. As a result, the cut must have shrunk to a point.

In general, the above integral equations should receive also contributions from string loop corrections [27, 28]. The 1-loop correction to the dressing phase considered in [28] produces extra contributions to the r.h.s. of the integral Bethe equation (4.18), but it is easy to see (e.g., from eq. (10) in [28]) that it is negligible in the present limit. This implies that the predictions of the asymptotic “undressed” gauge theory Bethe ansatz of [9] and full string Bethe ansatz should agree in this limit.¹⁸

4.3 Giant magnons and their bound states as finite-gap solutions

Let us now consider some simple solutions of equations (4.16) and (4.17) in the infinite J_2 limit. We start with solutions made up only of condensates and no cuts \mathcal{C}_k . Without cuts we can disregard eq. (4.17) and the condensates, whose contribution to the energy, spins and string momentum is additive, can be treated individually. The periodicity of the closed string forces the total string momentum to be an integer multiple of 2π . However, the momentum p from an individual condensate need not satisfy this condition as long as the total momentum coming from all the condensates that make up the closed string solution does satisfy the condition.

Hence, we may formally consider the case of a single condensate only, remembering that the final physical closed string solution will be made up of more than one condensate.

¹⁸This conclusion is consistent with the discussion in [3] where bound states of giant magnons were interpreted as poles of BDS S -matrix; it was assumed that the dressing factor does not introduce new poles.

It is useful to introduce a different spectral parameter y which satisfies $y = x + \frac{g^2}{2x}$ [9]. Then the equations on ρ in (4.16) become

$$\begin{aligned} \int_{\mathcal{B}} dy \rho(y) &= J_1, \\ \int_{\mathcal{B}} dy \frac{\rho(y)}{\sqrt{y^2 - 2g^2}} &= p, \\ 2g^2 \int_{\mathcal{B}} dy \frac{\rho(y)}{y\sqrt{y^2 - 2g^2} + y^2 - 2g^2} &= E - J_2 - J_1. \end{aligned} \tag{4.19}$$

In order for the momentum and the energy to be real we also require that the end points of the condensate be complex conjugate to each other. Assuming that $\rho(y) = -in$, we see from the first equation in (4.19) that the end points of the condensate are $y_0 \pm iJ_1/2$, where y_0 is to be determined. If we interpret $\rho(y)$ as a density of Bethe roots, then the contour would naturally be chosen to be a straight line along the imaginary direction in order that $dy\rho(y)$ is positive real. However, because of the square root in the second and third integral equations, there is a branch cut between $\pm\sqrt{2}g$ and so there is an ambiguity in how one chooses the contour. In particular, if we substitute this density into the second equation, we find the relation

$$\operatorname{arccosh}\left(\frac{y_0 + iJ_1/2}{\sqrt{2}g}\right) - \operatorname{arccosh}\left(\frac{y_0 - iJ_1/2}{\sqrt{2}g}\right) = i\frac{p}{n}, \tag{4.20}$$

where one can see a sign ambiguity in evaluating the arccosh. If we momentarily set $J_1 = 0$, then one can have the solution $y_0 = \sqrt{2}g \cos \frac{p}{2n}$, assuming that the end points are evaluated on opposite sides of the cut, which requires the contour to go outside one of the branch points. Otherwise, there is a solution only if p/n is a multiple of 2π . The more general solution is

$$y_0 = \sqrt{2g^2 \cos^2 \frac{p}{2n} + \left(\frac{J_1}{2n}\right)^2 \cot^2 \frac{p}{2n}}, \tag{4.21}$$

where one finds that a straight-line contour is possible if

$$J_1 > 2\sqrt{2}gn \sin \frac{p}{2n} \tan \frac{p}{2n}. \tag{4.22}$$

If we start with J_1 satisfying this bound and smoothly decrease the value, one will see that the contour starts deforming once J_1 is less than the bound. Even as $J_1 \rightarrow 0$, we are still left with a nontrivial contour. This is demonstrated in figure 2 where we show three contours with different values of J_1 and fixed p .

Finally, performing the final integral and putting in the value for y_0 in (4.21), one finds

$$E - J_2 = n\sqrt{\left(\frac{J_1}{n}\right)^2 + 8g^2 \sin^2 \frac{p}{2n}}, \tag{4.23}$$

which is the same as (2.17). The case with $n = 1$ corresponds to a single magnon with spin [3]. Other values of n represent bound states of n magnons with the string momentum p and S^3 angular momentum shared equally among magnons.

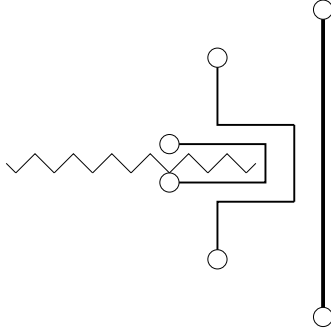


Figure 2: Condensates for three different values of J_1 . As $J_1 \rightarrow 0$, the end points of the contour approach the cut.

We can also derive similar relations directly from the discrete BDS Bethe equations. These equations for the Bethe roots y_j are [9]

$$\left(\frac{x(y_j + i/2)}{x(y_j - i/2)} \right)^{J_1 + J_2} = \prod_{k \neq j}^{J_1} \frac{y_j - y_k + i}{y_j - y_k - i} \quad (4.24)$$

where $x(y) = (y + \sqrt{y^2 - 2g^2})/2$. In the limit where $J_2 \rightarrow \infty$, there can be Bethe string solutions, where a string is made up of J_1 roots situated at $y_j = y_0 + i(J_1 + 1 - 2j)/2$ with $j = 1, \dots, J_1$ and y_0 real. The momentum contribution of a root satisfies

$$e^{ip_j} = \frac{x(y_j + i/2)}{x(y_j - i/2)}, \quad (4.25)$$

and so the total momentum coming from a Bethe string is

$$\begin{aligned} ip &= \sum_{j=1}^{J_1} \left[\ln[x(y_0 + i(J_1 + 2 - 2j)/2)] - \ln[x(y_0 + i(J_1 - 2j)/2)] \right] \\ &= \ln[x(y_0 + iJ_1/2)] - \ln[x(y_0 - iJ_1/2)] \\ &= \operatorname{arccosh} \left(\frac{y_0 + iJ_1/2}{\sqrt{2}g} \right) - \operatorname{arccosh} \left(\frac{y_0 - iJ_1/2}{\sqrt{2}g} \right), \end{aligned} \quad (4.26)$$

which matches (4.20) when $n = 1$. Likewise, $E - J_2 - J_1$ is [9]

$$\begin{aligned} E - J_2 - J_1 &= ig^2 \sum_{j=1}^{J_1} \left(\frac{1}{x(y_j + i/2)} - \frac{1}{x(y_j - i/2)} \right) \\ &= ig^2 \left(\frac{1}{x(y_0 + iJ_1/2)} - \frac{1}{x(y_0 - iJ_1/2)} \right). \end{aligned} \quad (4.27)$$

It is straightforward to show that this is the result for the third integral in (4.19) when $\rho = -i$; thus we get (4.23) with $n = 1$. More general values of n are obtained by increasing the density of the roots.

In appendix C we will derive an analogous equation for the $SL(2)$ sector. The other rank one sector, the $SU(1|1)$ sector, which is equivalent to free fermions in the one-loop approximation, does not have the poles and zeros in its S-matrix [42, 43] to build up (bound states of) giant magnons.

4.4 Finite-gap solutions for large spin limits of circular and pulsating strings

One interesting application of this discussion is a limit of the circular string solution of [22] considered already in section 2.3. Here we have $nJ_1 = mJ_2$, so that $J_2 \rightarrow \infty$ with finite J_1 corresponds to holding m fixed as $n \rightarrow \infty$. In [12, 8] it was argued that these solutions correspond to single-cut configurations and so $G(x)$ is an algebraic function

$$G(x) = \frac{L}{4} \left(\frac{1}{x - \frac{g}{\sqrt{2}}} + \frac{1}{x + \frac{g}{\sqrt{2}}} \right) + \frac{L}{4} \left[\frac{(1+\epsilon)^{-1/2}}{x - \frac{g}{\sqrt{2}}} + \frac{(1-\epsilon)^{-1/2}}{x + \frac{g}{\sqrt{2}}} \right] \sqrt{ax^2 + bx + c} - \pi n, \quad (4.28)$$

where $L = J_1 + J_2$ and with ϵ, a, b and c to be determined. In order to cancel the poles, a, b and c must satisfy $1 = \frac{g^2}{2} a + c$, $b = \frac{\sqrt{2}}{g} \epsilon$ while matching the asymptotics gives

$$\pi n = \frac{L\sqrt{a}}{4} \left(\frac{1}{\sqrt{1+\epsilon}} + \frac{1}{\sqrt{1-\epsilon}} \right), \quad \pi(n - 2m) = \frac{\sqrt{2}L\sqrt{c}}{4g} \left(\frac{1}{\sqrt{1+\epsilon}} - \frac{1}{\sqrt{1-\epsilon}} \right)$$

In the limit $n \rightarrow \infty$, one finds

$$\begin{aligned} \epsilon &= \frac{\sqrt{\lambda m}}{\sqrt{J_1^2 + m^2 \lambda}}, & b &= \frac{4\pi m}{\sqrt{J_1^2 + m^2 \lambda}}, \\ a &= \frac{1}{2} \frac{(4\pi m)^2}{J_1^2 + m^2 \lambda + J_1 \sqrt{J_1^2 + m^2 \lambda}}, & c &= \frac{1}{2} \frac{m^2 \lambda}{J_1^2 + m^2 \lambda + J_1 \sqrt{J_1^2 + m^2 \lambda}}, \end{aligned} \quad (4.29)$$

where we used the fact that $L/n = J_1/m$ in the limit when L and n both approach ∞ . In this limit the cut shrinks to a point with support at $x = x_0$, where

$$x_0 = \frac{1}{4\pi m} \left(\sqrt{J_1^2 + m^2 \lambda} + J_1 \right), \quad \text{i.e.} \quad y_0 = x_0 + \frac{g^2}{2x_0} = \frac{1}{2\pi m} \sqrt{J_1^2 + m^2 \lambda}. \quad (4.30)$$

As the cut shrinks to zero length, the density approaches $\rho(y) = J_1 \delta(y - y_0)$ and so $E - J_2$ approaches the same value as in (2.49) (with k in (2.49) replaced by m in the notation of the present section)

$$E - J_2 = J_1 + 2g^2 \int dy \frac{J_1 \delta(y - y_0)}{y \sqrt{y^2 - 2g^2} + y^2 - 2g^2} = \sqrt{J_1^2 + m^2 \lambda}. \quad (4.31)$$

Note that (4.30) and (4.31) are precisely the limiting values of, respectively, (4.21) and (4.23) in the limit $n \rightarrow \infty$ if $p = 2\pi m$. In other words, this limit of the circular string can be interpreted as a bound state of n magnons with each magnon having $1/n$ of the total energy and momentum.

One can also give a similar interpretation to the limit of pulsating string solutions discussed in [35, 8, 36]. The corresponding state is outside the $SU(2)$ sector on the gauge side but is still described by finite gap equations for a string on $R \times S^3$. We can write the ansatz for the pulsating string solution in terms of the complex coordinates X_1 and X_2 as (cf. (2.38))

$$X_1 = \sin \theta e^{im\sigma}, \quad X_2 = \cos \theta e^{i\varphi}, \quad \theta = \theta(\tau), \quad \varphi = \varphi(\tau). \quad (4.32)$$

This ansatz corresponds to a circular string wrapped m times and with its center of mass moving along the φ direction with momentum J , and which is pulsating back and forth along θ . The string equations of motion lead to

$$\dot{\varphi} = \frac{J}{\sqrt{\lambda} \cos^2 \theta}, \tag{4.33}$$

which applied to the conformal constraint gives

$$\kappa^2 = \dot{\theta}^2 + m^2 \sin^2 \theta + \frac{J^2}{\lambda \cos^2 \theta}. \tag{4.34}$$

If we now assume that $J/\sqrt{\lambda} \gg 1$, and $m \gg 1$ with m/J fixed, and further assume that $\theta \ll 1$, the pulsating becomes harmonic and the constraint equation (4.34) is well approximated by

$$\frac{E^2 - J^2}{\lambda} = \dot{\theta}^2 + \left(m^2 + \frac{J^2}{\lambda}\right) \theta^2. \tag{4.35}$$

Further assuming that $E - J$ is held fixed and following the analogy with the standard harmonic oscillator quantization ($\epsilon = \hbar\omega N$ where here $\omega^2 = m^2 + \frac{J^2}{\lambda}$) we find that

$$E - J \approx \sqrt{\left(\frac{mN}{J}\right)^2 \lambda + N^2}, \tag{4.36}$$

where N is the oscillator mode number which must satisfy $N \ll J$ in order that $\theta \ll 1$.

The result (4.36) can also be reproduced from solutions of the finite gap equation in [8]. In [8] it was shown that the resolvent arising from the pulsating solution is

$$G(x) = \frac{1}{2} \frac{1}{x^2 - g^2/2} \left(Ex + \sqrt{[2\pi m(x^2 - g^2/2) - Jx]^2 + (E^2 - J^2)x^2} \right) - \pi m. \tag{4.37}$$

This resolvent clearly has four branch points and two cuts. If we now take the limit $E, J \rightarrow \infty$ with $E - J$ and J/m finite, then the two branch cuts each shrink to a point at

$$x = \frac{1}{4\pi} \left[\frac{J}{m} \pm \sqrt{\left(\frac{J}{m}\right)^2 + \lambda} \right], \quad \text{i.e.} \quad y = \pm \frac{1}{2\pi} \sqrt{\left(\frac{J}{m}\right)^2 + \lambda} = \pm y_0. \tag{4.38}$$

Hence, the solution has reduced to two zero length condensates which are images of each other. The densities along the condensates are opposite to each other so that $J_1 = 0$. Each condensate contributes half the oscillator number, so

$$\rho(y) = \frac{N}{2} \left(\delta(y - y_0) - \delta(y + y_0) \right). \tag{4.39}$$

The total momentum in (4.19) must be zero, which means we should choose the branches $\sqrt{(\pm y_0)^2 - 2g^2} > 0$. Finally, the third equation in (4.19) leads to

$$\begin{aligned} E - J &= \frac{\lambda}{4\pi} \left(\frac{N}{\left(\frac{J}{m}\right)^2 + \frac{J}{m} \sqrt{\left(\frac{J}{m}\right)^2 + \lambda}} - \frac{N}{\left(\frac{J}{m}\right)^2 - \frac{J}{m} \sqrt{\left(\frac{J}{m}\right)^2 + \lambda}} \right) \\ &= N \sqrt{\left(\frac{m}{J}\right)^2 + \lambda}, \end{aligned} \tag{4.40}$$

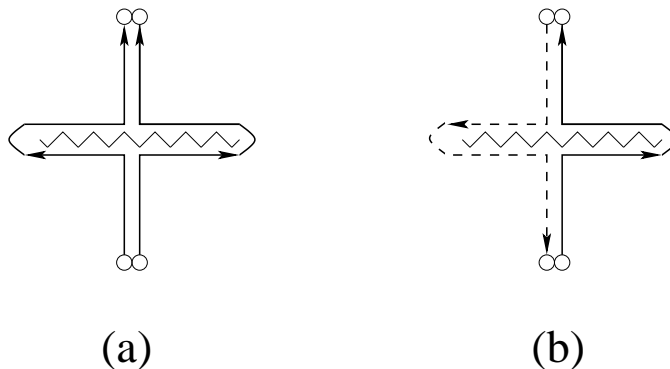


Figure 3: Condensates for strings made up of two giant magnons. (a) is the limit of the folded string and (b) is a pulsating string. The arrows represent the sign of the density while the dashed line in (b) indicates that the condensate is on the lower sheet.

reproducing (4.36).

We can also work backward and find giant magnon solutions in the pulsating sector. These solutions would correspond to condensates of equal length and opposite density with total oscillator number $N/2$ on each condensate. If the density is given by $\pm n$ on each condensate, then the computation goes through exactly as for the $SU(2)$ case, but with J_1 replaced by $N/2$ and J_2 by J . The two condensates have momentum $\pm p$, so one finds

$$E - J = 2n \sqrt{\left(\frac{N}{2n}\right)^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2n}}. \quad (4.41)$$

We can reduce this to (4.36) by taking $n \rightarrow \infty$ and identifying $p = mN\pi/J$.

In figure 3 we show the contours for (a) the limit of the folded string and (b) the analogous configuration for a pulsating string. The distinction between these two cases is that the folded string has both condensates on the same sheet, while the pulsating string has its condensates on different sheets. The string motion in (b) can be viewed as follows: for half the string, say from $0 < \sigma < \pi$, the configuration is exactly the same as the limit of the folded string, with the string having constant angular velocity along φ_1 . On the other half of the string everything is the same, except the angular velocity along φ_1 is in the opposite direction. Even though the separate halves are rotating in opposite directions in φ_1 , the string is continuous since the two halves are attached where $\cos \theta = 0$. Thus, the string oscillates between a folded configuration² and a circular configuration twice every revolution in φ_1 .

In appendices B and C we shall also discuss similar solutions in the $SL(2)$ sector.

While this paper was in preparation we learned of an interesting forthcoming paper [41] that discusses the finite J generalization of the giant magnon solutions of [2, 3].

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A. Fluctuation Lagrangian near the folded string solution

A.1 Bosonic fluctuations

Our starting point will be the general form of the 2-spin folded string solution discussed in section 2.2. We shall consider the conformal gauge.

Since the string is not stretched in the spatial AdS_5 directions (with the metric $ds^2 = -\frac{(1+\frac{1}{4}\zeta^2)^2}{(1-\frac{1}{4}\zeta^2)^2}dt^2 + \frac{d\zeta_k d\zeta_k}{(1-\frac{1}{4}\zeta^2)^2}$) their fluctuations $t = \kappa\tau + \tilde{t}$, $\zeta_k = 0 + \tilde{\zeta}_k$, $k = 1, 2, 3, 4$ are governed by

$$L = -\frac{1}{2}[-(\partial_a \tilde{t})^2 + (\partial_a \tilde{\zeta}_k)^2 + \kappa^2 \tilde{\zeta}_k^2], \quad (4.1)$$

i.e. we get one massless fluctuation and 4 massive ones with the characteristic frequencies $\omega = \pm\sqrt{n^2 + \kappa^2}$.

To consider the S^5 fluctuations we shall follow [22, 37] and use complex embedding coordinates in terms of which the S^5 Lagrangian is

$$L = -\frac{1}{2}\partial_a X_i \partial^a X_i^* + \frac{1}{2}\Lambda(X_i X_i^* - 1), \quad (4.2)$$

and the classical solution is

$$X_1 = \cos \theta(\sigma) e^{iw_1\tau}, \quad X_2 = \sin \theta(\sigma) e^{iw_2\tau}, \quad X_3 = 0, \quad (4.3)$$

so that the classical value of the Lagrange multiplier is

$$\Lambda = \partial_a X_i \partial^a X_i^* = -2(\kappa^2 - w_1^2) \frac{\sin^2 \theta}{\sin^2 \theta_*} - 2w_1^2 + \kappa^2. \quad (4.4)$$

Introducing the fluctuations $X_i \rightarrow X_i + \tilde{X}_i$ one gets

$$\tilde{L} = -\frac{1}{2}\partial_a \tilde{X}_i \partial^a \tilde{X}_i^* + \frac{1}{2}\Lambda \tilde{X}_i \tilde{X}_i^*, \quad \sum_{i=1}^3 (X_i \tilde{X}_i^* + X_i^* \tilde{X}_i) = 0. \quad (4.5)$$

\tilde{X}_3 has no classical background and thus decouples, i.e. its equation of motion is

$$\partial_0^2 \tilde{X}_3 - \partial_1^2 \tilde{X}_3 - \Lambda \tilde{X}_3 = 0, \quad (4.6)$$

where $\Lambda = \Lambda(\sigma)$.

The remaining 3 independent fluctuations are coupled. Let us define

$$\tilde{X}_1 = e^{iw_1\tau}(g_1 + if_1), \quad \tilde{X}_2 = e^{iw_2\tau}(g_2 + if_2), \quad (4.7)$$

where the constraint in (4.5) implies

$$g_1 \cos \theta + g_2 \sin \theta = 0. \quad (4.8)$$

Then

$$\begin{aligned} \tilde{L} = \frac{1}{2} & \left[\dot{g}_1^2 + \dot{g}_2^2 + \dot{f}_1^2 + \dot{f}_2^2 - g_1'^2 - g_2'^2 - f_1'^2 - f_2'^2 + w_1^2(f_1^2 + g_1^2) + w_2^2(g_2^2 + f_2^2) \right. \\ & \left. - 4w_1 f_1 \dot{g}_1 - 4w_2 f_2 \dot{g}_2 + \Lambda(f_1^2 + f_2^2 + g_1^2 + g_2^2) \right] \end{aligned} \quad (4.9)$$

We can simplify this by introducing

$$\xi = g_1 \cos \theta + g_2 \sin \theta, \quad \eta = -g_1 \sin \theta + g_2 \cos \theta, \quad (4.10)$$

and (4.8) implies that $\eta_1 = 0$. The fluctuation Lagrangian for f_1, f_2, η then becomes

$$\begin{aligned} \tilde{L} = \frac{1}{2} & \left[\dot{f}_1^2 + \dot{f}_2^2 - f_1'^2 - f_2'^2 + \dot{\eta}^2 - \eta'^2 - M_\eta^2 \eta^2 - M_1^2 f_1^2 - M_2^2 f_2^2 \right. \\ & \left. + 4(w_1 \sin \theta f_1 - \frac{\sqrt{\kappa^2 - w_1^2 \cos^2 \theta_*}}{\sin \theta_*} \cos \theta f_2) \dot{\eta} \right], \end{aligned} \quad (4.11)$$

where

$$M_\eta^2 = -(\kappa^2 - w_1^2) \frac{\cos 2\theta}{\sin^2 \theta_*}, \quad (4.12)$$

$$M_1^2 = -(\kappa^2 - w_1^2) \left(1 - 2 \frac{\sin^2 \theta}{\sin^2 \theta_*} \right), \quad M_2^2 = -(\kappa^2 - w_1^2) \left(1 + \frac{\cos 2\theta}{\sin^2 \theta_*} \right), \quad (4.13)$$

and we used the explicit form of w_2 from (2.24).

A.2 Fermionic fluctuations

The quadratic part of the $AdS_5 \times S^5$ superstring Lagrangian evaluated on a bosonic solution has a simple form (see [7, 13, 5, 23] for details)

$$L_F = i \left(\eta^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ} \right) \bar{\vartheta}^I \rho_a D_b \vartheta^J, \quad \rho_a \equiv \Gamma_A e_a^A, \quad e_a^A \equiv E_\mu^A(\mathcal{X}) \partial_a \mathcal{X}^\mu, \quad (4.14)$$

where $I, J = 1, 2$, $s^{IJ} = \text{diag}(1, -1)$, ρ_a are projections of the ten-dimensional Dirac matrices and \mathcal{X}^μ are the coordinates of the AdS_5 space for $\mu = 0, 1, 2, 3, 4$ and the coordinates of S^5 for $\mu = 5, 6, 7, 8, 9$. The covariant derivative is given by

$$D_a \vartheta^I = \left(\delta^{IJ} D_a - \frac{i}{2} \epsilon^{IJ} \Gamma_* \rho_a \right) \vartheta^J, \quad \Gamma_* \equiv i \Gamma_{01234}, \quad \Gamma_*^2 = 1, \quad (4.15)$$

where $D_a = \partial_a + \frac{1}{4} \omega_a^{AB} \Gamma_{AB}$, $\omega_a^{AB} \equiv \partial_a \mathcal{X}^\mu \omega_\mu^{AB}$. Fixing the κ -symmetry by the same condition as in [23] $\vartheta^1 = \vartheta^2 = \vartheta$ one gets

$$L_F = -2i \bar{\vartheta} D_F \vartheta, \quad D_F = -\rho^a D_a - \frac{i}{2} \epsilon^{ab} \rho_a \Gamma_* \rho_b. \quad (4.16)$$

Labelling the coordinates as follows:

$$\begin{aligned} \mu : & 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \\ \mathcal{X}^\mu : & t \ \rho \ \psi \ \phi_1 \ \phi_2 \ \gamma \ \varphi_3 \ \theta \ \varphi_1 \ \varphi_2 \end{aligned} \quad (4.17)$$

we find that in the case of the folded solution that the non-trivial components of the Lorentz connection ω_a^{AB} are

$$\omega_0^{87} = -w_1 \sin \theta, \quad \omega_0^{97} = w_2 \cos \theta. \quad (4.18)$$

For ρ_a we find

$$\rho_0 = \kappa\Gamma_0 + w_1 \cos \theta\Gamma_8 + w_2 \sin \theta\Gamma_9, \quad \rho_1 = \Gamma_7\theta' \quad (4.19)$$

The operator D_F becomes

$$\begin{aligned} D_F &= (\kappa\Gamma_0 + w_1 \cos \theta\Gamma_8 + w_2 \sin \theta\Gamma_9)\partial_0 - \Gamma_7\theta'\partial_1 \\ &\quad - \frac{1}{2}(\kappa\Gamma_0 + w_1 \cos \theta\Gamma_8 + w_2 \sin \theta\Gamma_9)(w_1 \sin \theta\Gamma_{87} - w_2 \cos \theta\Gamma_{97}) \\ &\quad + \theta'(w_1 \cos \theta\Gamma_8 + w_2 \sin \theta\Gamma_9)\Gamma_{07}\Gamma_{1234} \end{aligned} \quad (4.20)$$

In section 3.1 we shall consider the special limit of this operator when $\theta_* = \frac{\pi}{2}$ and $\kappa \rightarrow \infty$.

A.3 Some details

In the main text we also use the general expression for the 1-loop correction to the energy in terms of the bosonic and fermionic characteristic frequencies [37]

$$E_1 = \frac{1}{\kappa}E_{2d} = \frac{1}{2\kappa} \left[\sum_{p=1}^8 (\hat{w}_{p,0}^B - \hat{w}_{p,0}^F) + \sum_{n=1}^{\infty} \sum_{I=1}^{16} (\hat{w}_{I,n}^B - \hat{w}_{I,n}^F) \right], \quad (4.21)$$

$$\hat{w}_{p,0} = \text{sign}(C_p^B)\omega_{p,0}, \quad \hat{w}_{I,n} = \text{sign}(C_{I,B}^{(n)})\omega_{I,n}, \quad (4.22)$$

$$C_p^B = \frac{1}{2m_{11}(\omega_{p,0})\omega_{p,0} \prod_{q \neq p} (\omega_{p,0}^2 - \omega_{q,0}^2)}, \quad C_{I,B}^{(n)} = \frac{1}{m_{11}(\omega_{I,n}) \prod_{J \neq I} (\omega_{I,n} - \omega_{J,n})}, \quad (4.23)$$

where m_{11} is a minor of F , i.e. the determinant of the matrix obtained from F by removing the first row and first column, with F being the matrix entering the equation $\det F = 0$ for the characteristic frequencies. This matrix satisfies the condition $F^T(\omega_{I,n}, n) = F(-\omega_{I,n}, -n)$ (see [37] for details).

Let us also explain how one arrives at eq.(3.9) of section 3.1, and, in particular, why one can indeed ignore the contribution of the $x = 0$ point. From (3.8) we get

$$\ddot{f}_1 - f_1'' - (w^2 - 1)f_1 + 2\epsilon(x)w\dot{\eta} = 0, \quad (4.24)$$

$$\ddot{\eta} - \eta'' - (w^2 - 1)\eta - 2\epsilon(x)w\dot{f}_1 = 0, \quad (4.25)$$

and looking for solutions $f_1 \sim A(x)e^{i\omega t}$, $\eta \sim B(x)e^{i\omega t}$ we get

$$A'' + (\omega^2 + w^2 - 1)A - 2i\omega w\epsilon(x)B = 0, \quad B'' + (\omega^2 + w^2 - 1)B + 2i\omega w\epsilon(x)A = 0 \quad (4.26)$$

Combining these two equations we get a 4-th order differential equation for A , which (after using that $\delta(x)\epsilon(x) = 0$) becomes

$$\begin{aligned} \epsilon^2(x)[A'''' + \omega^2 A'' + (w^2 - 1)A'] - 4\epsilon^4(x)\omega^2 w^2 A + \delta(x)[\omega^2 A + A'' + (w^2 - 1)A] \\ + \epsilon^2(x)(\omega^2 + w^2 - 1)[\omega^2 A + A'' + (w^2 - 1)A] = 0 \end{aligned} \quad (4.27)$$

We can solve this equation for $x < 0$ and $x > 0$ with the ansatz $A \sim e^{ipx}$ and obtain the characteristic frequencies (3.9). Notice that the equation (4.27) contains a delta-function term which signals a discontinuity at the origin. Integrating (4.27) near $x = 0$ and taking the interval of integration to zero we find that the only non-vanishing term is

$$\omega^2 A(0) + A''(0) + (w^2 - 1)A(0) = 0 . \tag{4.28}$$

One can see that one cannot have the solution $A \sim e^{ipx}$ valid at the origin since the frequencies (3.9) do not satisfy equation (4.28) unless $w = 0$. To satisfy (4.28) also for $w \neq 0$ we need to have $A(0) = 0$. This shows that $A(x)$ is discontinuous at origin. Therefore, one can just ignore the $x = 0$ point and thus obtain (3.9).

B. Large J limit of circular (S, J) solution in the $SL(2)$ sector

It is straightforward to perform the analog of the analysis of sections 2.3 and 3.2 and consider the $J \gg S$ limit of the circular 2-spin solution in the $SL(2)$ sector [22, 37]. One finds again the square root formula for the classical energy similar to (2.46) and also that 1-loop correction to it vanishes.

B.1 Limit of classical solution

Let us start with a review of the solution [22, 37] describing circular string which is rotating both in AdS_5 and in S^5 . In terms of complex combination of embedding coordinates one has

$$Y_0 = r_0 e^{ik\tau} , \quad Y_1 = r_1 e^{iw\tau + im\sigma} , \quad X_1 = e^{iw\tau + ik\sigma} , \quad Y_2, X_2, X_3 = 0 \tag{B.1}$$

$$r_0 \equiv \cosh \rho_0 , \quad r_1 \equiv \sinh \rho_0 , \quad r_0^2 - r_1^2 = 1 . \tag{B.2}$$

Here ρ_0 is a constant radius of the circular string in AdS_5 , k and m are the winding numbers, and w and w are rotation frequencies of the string. From equations of motion we have

$$w^2 = \kappa^2 + m^2 , \quad w^2 = \nu^2 + k^2 , \quad \nu^2 = -\Lambda , \quad \kappa^2 = \tilde{\Lambda} , \tag{B.3}$$

where Λ and $\tilde{\Lambda}$ are the Lagrange multipliers for the embedding coordinates. The energy and the two non-zero spins are

$$E = \sqrt{\lambda} \mathcal{E} = \sqrt{\lambda} r_0^2 \kappa , \quad S = \sqrt{\lambda} \mathcal{S} = \sqrt{\lambda} r_1^2 w , \quad J = \sqrt{\lambda} \mathcal{J} = \sqrt{\lambda} w , \tag{B.4}$$

and the conformal gauge constraints imply

$$2\kappa \mathcal{E} - \kappa^2 = 2\sqrt{\kappa^2 + m^2} \mathcal{S} + \mathcal{J}^2 + k^2 , \tag{B.5}$$

$$m \mathcal{S} + k \mathcal{J} = 0 , \tag{B.6}$$

while (B.2) gives also

$$\frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}}{\sqrt{m^2 + \kappa^2}} = 1 . \tag{B.7}$$

Eliminating κ from (B.5) and (B.7) one finds $\mathcal{E} = \mathcal{E}(\mathcal{S}, \mathcal{J}, m)$.

Let us now consider the special limit when $\mathcal{J} \rightarrow \infty$ with \mathcal{S} and k being fixed and negative (this implies $m \gg 1$). Then \mathcal{E} is also divergent but $\mathcal{E} - \mathcal{J}$ is finite. We get

$$\kappa = \frac{\mathcal{J}}{|k|} + \frac{k^2}{\sqrt{k^2 + \mathcal{S}^2}} + O\left(\frac{1}{\mathcal{J}}\right), \quad (\text{B.8})$$

$$r_0 = 1 + \frac{\mathcal{S}^2}{2\sqrt{k^2 + \mathcal{S}^2}} \frac{1}{\mathcal{J}} + \dots, \quad r_1 = \frac{\mathcal{S}}{(k^2 + \mathcal{S}^2)^{1/4}} \frac{1}{\sqrt{\mathcal{J}}} + \dots, \quad (\text{B.9})$$

$$w = \frac{\mathcal{J}}{\mathcal{S}} \sqrt{k^2 + \mathcal{S}^2} + \frac{\mathcal{S}k^2}{k^2 + \mathcal{S}^2} + \dots \quad (\text{B.10})$$

and finally in the limit of $\mathcal{J} \rightarrow \infty$

$$E - J = \sqrt{\mathcal{S}^2 + k^2 \lambda}. \quad (\text{B.11})$$

B.2 Vanishing of 1-loop correction to classical energy

Let us set $k = -1$ for simplicity. For generic \mathcal{J} and \mathcal{S} the bosonic and fermionic fluctuation frequencies were obtained in [37]. There are 4 real free massive fields with mass ν , for which in the limit (and after the rescaling of the coordinates $t = \kappa\tau$, $x = \kappa\sigma$) we get $\omega = \pm\sqrt{p^2 + 1}$. There are also two free massive modes with mass κ , which in the limit has the same frequencies. The remaining coupled fluctuation Lagrangian in the large m -limit reads (cf. (3.4))

$$\bar{L} = \frac{1}{2} \left(\dot{f}_1^2 - f_1'^2 + \dot{F}_0^2 - F_0'^2 + \dot{F}_1^2 - F_1'^2 + \dot{G}_1^2 - G_1'^2 \right) - 2\sqrt{1 + \mathcal{S}^{-2}} F_1 \dot{G}_1 + 2F_1 G_1', \quad (\text{B.12})$$

where F_0 , F_1 and G_1 are fluctuations in AdS_5 directions. The non-trivial characteristic frequencies are found to be similar to the ones in the $SU(2)$ case (cf. (3.22))

$$\omega_{1,2} = \sqrt{1 + \beta^2} \pm \sqrt{(p + \beta)^2 + 1}, \quad \omega_{3,4} = -\sqrt{1 + \beta^2} \pm \sqrt{(p - \beta)^2 + 1}, \quad (\text{B.13})$$

$$\beta \equiv \mathcal{S}^{-1}.$$

The fermionic fluctuation Lagrangian has the following general form [37]

$$L = 2i\bar{\vartheta} D_F \vartheta, \quad D_F = \Gamma_0 \partial_0 - \Gamma_3 \partial_1 \pm ia\Gamma_1 + c\Gamma_{016} + d\Gamma_{136}, \quad (\text{B.14})$$

where

$$a = \frac{\sqrt{2}m\kappa r_0 r_1}{\sqrt{\kappa^2 - \nu^2}}, \quad c = \frac{\kappa k w^2 - w^2}{w \kappa^2 - \nu^2}, \quad d = \frac{km\kappa r_0^2}{\kappa^2 - \nu^2}. \quad (\text{B.15})$$

Expanding in large \mathcal{J} and rescaling the coordinates we obtain for $k = -1$

$$D_F = \Gamma_0 \partial_t - \Gamma_3 \partial_x \pm i\Gamma_1 - \frac{1}{2}\beta\Gamma_{016} - \sqrt{1 + \beta^2}\Gamma_{136}. \quad (\text{B.16})$$

The resulting fermionic characteristic frequencies are

$$\omega = \pm\sqrt{1 + \beta^2} \pm \sqrt{\left(p \pm \frac{1}{2}\beta\right)^2 + 1}. \quad (\text{B.17})$$

Proceeding as in the $SU(2)$ sector in section 3.2 to compute the 1-loop correction to the energy, we again find using (4.23) that in both the bosonic and fermionic cases the p -independent square roots in (B.13) and (B.17) do not contribute to E_1 . As a result, we get the same integral (3.29) as in the $SU(2)$ case

$$E_1 = \frac{1}{2} \int_{-\infty}^{\infty} dp \left[6\sqrt{p^2+1} + \sqrt{(p+\beta)^2+1} + \sqrt{(p-\beta)^2+1} - 4\sqrt{\left(p+\frac{1}{2}\beta\right)^2+1} - 4\sqrt{\left(p-\frac{1}{2}\beta\right)^2+1} \right] = 0. \quad (\text{B.18})$$

C. Giant magnons in the $SL(2)$ sector

In this appendix we shall consider “giant magnons” in the $SL(2)$ sector, i.e. the analogs of the solutions of [2] and of section 2.1 that have spins in both AdS_5 and S^5 . These “magnons” turn out to stretch to the boundary of AdS_5 and, strictly speaking, have not only infinite energy, but also infinite $E - J$. However, this infinity, unlike the usual infinity for E or J is associated with the boundary, and as such can be removed with a local counterterm. The final result is finite.

The setup is similar to the $SU(2)$ case in section 2.1. The relevant metric is that of $AdS_3 \times S^1$ part of $AdS_5 \times S^5$

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\chi^2 + d\phi^2, \quad (\text{B.1})$$

and we make the ansatz

$$\begin{aligned} t &= \tau, & \phi &= t + \varphi(\sigma) \\ \rho &= \rho(\sigma), & \chi &= w(t - \psi(\sigma)). \end{aligned} \quad (\text{B.2})$$

We then find that \mathcal{D} in the action (2.5) is given by

$$\begin{aligned} \mathcal{D} &= (\cosh^2 \rho - 1 - w^2 \sinh^2 \rho) (\partial_\sigma \varphi)^2 + w^2 \sinh^2 \rho (\partial_\sigma \psi)^2 + (\partial_\sigma \rho)^2 \\ &\quad + (\partial_\sigma \varphi - w^2 \sinh^2 \rho \partial_\sigma \psi)^2 \\ &= \cosh^2 \rho (\partial_\sigma \varphi)^2 + w^2 \sinh^2 \rho \cosh^2 \rho (\partial_\sigma \psi)^2 + (1 - w^2) \sinh^2 \rho (\partial_\sigma \rho)^2 \\ &\quad - w^2 \sinh^2 \rho (\partial_\sigma \varphi + \partial_\sigma \psi)^2. \end{aligned} \quad (\text{B.3})$$

The resulting equations of motion have the special solution for ψ

$$\partial_\sigma \psi = \frac{1}{\sinh^2 \rho} \partial_\sigma \varphi. \quad (\text{B.4})$$

Substituting it back into the action we have the same expression as in (2.9), except that now

$$r = \cosh \rho = \frac{\sin \varphi_0}{\sin \varphi}, \quad -\varphi_0 < \varphi < \varphi_0. \quad (\text{B.5})$$

The difference $E - J$ and the spin S are then given by

$$\begin{aligned} E - J &= \frac{\sqrt{\lambda}}{2\pi\sqrt{1-w^2}} \int_{-\varphi_0}^{\varphi_0} d\varphi \frac{\sin \varphi_0}{\sin^2 \varphi} \\ S &= w(E - J). \end{aligned} \quad (\text{B.6})$$

Strictly speaking, the quantities in (B.6) are infinite because of the singularity at $\varphi = 0$. This corresponds to $\rho = \infty$ which is at the boundary of AdS_5 . Hence, this divergence is in the UV and differs from the individual divergences of E and J which are in the IR. Accordingly, the divergence can be cancelled with a counterterm. This is accomplished by deforming the contour slightly away from $\varphi = 0$, giving the regulated answers

$$(E - J)_{\text{reg}} = -\frac{\sqrt{\lambda} \cos \varphi_0}{\pi \sqrt{1 - w^2}}, \quad S_{\text{reg}} = -\frac{w \sqrt{\lambda} \cos \varphi_0}{\pi \sqrt{1 - w^2}}, \quad (\text{B.7})$$

where the subscript (reg) refers to the regulated quantities. We can then write

$$(E - J)_{\text{reg}} = -\sqrt{|S_{\text{reg}}|^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}. \quad (\text{B.8})$$

One can also derive this result using the finite gap analysis. We first remark that an $SL(2)$ spin chain, strictly speaking, cannot have Bethe strings of finite size. For example, the Bethe equations for the one loop anomalous dimension in the $SL(2)$ sector are

$$\left(\frac{y_j - i/2}{y_j + i/2}\right)^J = \prod_{k \neq j}^S \frac{y_j - y_k + i}{y_j - y_k - i}. \quad (\text{B.9})$$

In the limit $J \rightarrow \infty$, the left hand side is zero if $\text{Im } y_j > 0$. This means that the right hand side must also be zero, which can be accomplished only if there is also a root at $y_j + i$. But then replacing by y_j by $y_j + i$ in the l.h.s. of (B.9) we again end up with a zero, which means that there is a root at $y_j + 2i$, and the argument continues *ad infinitum*. Hence, there are an infinite number of roots in the string and so S is infinite.

When taking the continuum limit, the Bethe equations turn into integral equations and the Bethe strings become condensates. In the finite gap equations this translates into condensates of infinite extent. Furthermore, in order for the energies to be real, every infinite condensate must be paired with its complex conjugate. The finite gap equations for the $SL(2)$ sector are very similar to the $SU(2)$ equations [38], and, in particular, the equations in (4.19) are the same with J_1 and J_2 replaced by S and J . Hence, we find that for an infinite condensate and its conjugate

$$S = \int_{-i\infty+y_0}^{+i\infty+y_0} dy \rho(y) - \int_{-i|S|/2+y_0}^{+i|S|/2+y_0} dy \rho(y). \quad (\text{B.10})$$

The first integral is infinite if $\rho = -i$ along the path. However, if we deform the contour slightly the integral will be zero, since ρ only has a double pole at infinity. Hence we find $S_{\text{reg}} = -|S_{\text{reg}}|$. Likewise,

$$(E - J)_{\text{reg}} = S_{\text{reg}} - 2g^2 \int_{-i|S|/2+y_0}^{+i|S|/2+y_0} dy \frac{\rho(y)}{y \sqrt{y^2 - 2g^2} + y^2 - 2g^2}. \quad (\text{B.11})$$

We solve for y_0 the same way as in section 4.3 and then (B.11) immediately gives (B.8).

The same result can be derived for the $SL(2)$ sector from the discrete asymptotic BDS-type Bethe equations in [42, 43]. The arguments work in almost the same way as for the $SU(2)$ sector as discussed in section 4. In this case the Bethe equations become

$$\left(\frac{x_j^-}{x_j^+}\right)^J = \prod_{k \neq j}^S \frac{y_j - y_k + i}{y_j - y_k - i} \left(\frac{1 - \frac{g^2}{2x_j^- x_k^+}}{1 - \frac{g^2}{2x_j^+ x_k^-}}\right)^2, \quad (\text{B.12})$$

where $x_j^\pm = x(y_j \pm i/2)$. Hence, as in the one-loop case if $\text{Im } y_j > 0$, then there must be a root at $y_j + i$. Hence, the Bethe string goes on forever in the imaginary direction. In order to have real solutions, we require that there also be the complex conjugate of this Bethe string. In any case, one now finds that

$$\begin{aligned} (E - J)_{\text{reg}} - S_{\text{reg}} &= ig^2 \sum_{j=1}^{\infty} \left(\frac{1}{x(y_0 + i|S_{\text{reg}}|/2 + ij)} - \frac{1}{x(y_0 + i|S_{\text{reg}}|/2 + ij - i)} \right) \\ &\quad + ig^2 \sum_{j=1}^{\infty} \left(\frac{1}{x(y_0 - i|S_{\text{reg}}|/2 - ij + i)} - \frac{1}{x(y_0 - i|S_{\text{reg}}|/2 + ij)} \right) \\ &= -ig^2 \left(\frac{1}{x(y_0 + i|S_{\text{reg}}|/2)} - \frac{1}{x(y_0 - i|S_{\text{reg}}|/2)} \right). \end{aligned} \quad (\text{B.13})$$

This then leads to (B.8).

The negative sign in front of the square root in (B.8) may seem puzzling, so let us try to give a possible interpretation of this configuration on the gauge side. The divergence of S and $E - J$ is due to the string going out to the boundary of AdS_5 . This suggests that we have inserted a localized adjoint gauge source, in other words, a Wilson line in the adjoint representation along a particular trajectory of the gauge theory. The infinite value for $E - J$ can then be interpreted as the infinite contribution coming from a source of infinite mass, as was the case for the quark-antiquark configuration in [39, 40]. Likewise, if the source is moving along the boundary, it will have infinite angular momentum if it has infinite mass. The regularization then corresponds to subtracting off this infinite energy and angular momentum and the resulting finite $E - J$ and S are the contributions of the operators in the presence of these sources. If one thinks of the boundary theory as being defined on $R \times S^3$, then the allowed states must be color singlets on S^3 . Hence, if an adjoint source is inserted somewhere on the S^3 , this must bind onto states such that the net color is zero.¹⁹ With the background color source, we see no violation of the usual supersymmetry arguments that normally enforce $E \geq J$.

¹⁹Let us note that in the Poincare coordinates in AdS_5 with the metric $ds^2 = \frac{R^2}{z^2}(-dt^2 + dr^2 + r^2 d\theta^2 + dz^2)$, the above solution has the form:

$$z = \frac{R \sin \varphi}{\sin \varphi_0 \cos t}, \quad t = R \tan t, \quad r^2 = (R^2 + t^2) \left(1 - \frac{\sin^2 \varphi}{\sin^2 \varphi_0}\right), \quad \theta = w \arccos\left(\frac{R}{\sqrt{R^2 + t^2}}\right).$$

The boundary is at $z = 0$ which occurs at $\varphi = 0$. Here t is the global time and t refers to the Poincare patch time. The trajectory at the boundary has the source coming in from infinity and reaching a minimum distance R at $t = 0$. In the meantime its angle changes between $-\frac{w\pi}{2}$ and $+\frac{w\pi}{2}$ (as w approaches 1 the trajectory approaches a lightlike straight line).

Note that the circular $SL(2)$ solution discussed in appendix B is *not* made up of magnons of this type. Instead, the circular solution has a single cut shrinking to zero size along the real axis, which contrasts with the $SU(2)$ case where it is a cut along the imaginary direction that is shrinking. But the bound magnons correspond to roots extended along the imaginary direction, and so, unlike the $SU(2)$ case, it is not possible to see the $SL(2)$ circular solution emerging as a limiting case of bound magnons.

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